On solving the multi-period single-sourcing problem under uncertainty*

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Abstract

We present a framework for solving the strategic problem of assigning retailers to facilities in a multi-period single-sourcing product environment under uncertainty in the demand from the retailers and the cost of production, inventory holding, backlogging and distribution of the product. By considering a *splitting variable* mathematical representation of the *Deterministic Equivalent Model*, we specialize the so-called *Branch-and-Fix Coordination* algorithmic framework. It exploits the structure of the model and, specifically, the *non-anticipativity* constraints for the assignment variables. The algorithm uses the *Twin Node Family* (*TNF*) concept. Our procedure is specifically designed for coordinating the selection of the branching *TNF* and the branching *S3* set, such that the *non-anticipativity* constraints are satisfied. Some computational experience is reported.

Keywords: two-stage stochastic mixed 0–1 programs, non-anticipativity constraints, splitting variables, twin node family, branch-and-fix coordination, fix-and-relax coordination.

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1 Introduction

Given a planning horizon, a set of retailers and a set of facilities (e.g., production plants), the *Multi-Period Single-Sourcing Problem* (*MPSSP*) is concerned with assigning each retailer to a unique facility at the beginning of the planning horizon. The aim is to minimize the assignment, inventory holding and backlogging costs subject to the satisfaction of retailers' demands and the production capacity constraints at the facilities. The assignment cost includes the production and distribution costs. The problem can be viewed as an assignment problem where the goodness of the retailers' assignment can be measured against its performance along the planning horizon.

The deterministic version of the problem, see [15], where it is assumed that the demand and the costs related parameters are known, is *NP-hard*. Moreover, very frequently these parameters are uncertain. Therefore, the *MPSSP* is an interesting application case of Stochastic Integer Programming (*SIP*).

The stochastic approaches for production planning usually only consider tactical decisions (modelled by continuous variables, see e.g. [12]). Moreover, there are few schemes that address the strategic production planning under uncertainty (modelled by using 0–1 as well as continuous variables), see [2, 4, 6, 19, 20, 21, 22], among others. Most of these approaches only consider mean (expected) objective functions. Alternatively, very few approaches deal with the mean-risk measures by considering semi-deviations [23] and excess probabilities [25]. These approaches are more amenable for large-scale problem solving than the classical mean-variance schemes, mainly in the presence of 0–1 variables. See also [26].

In this paper we present a mixed 0–1 *Deterministic Equivalent Model* (*DEM*) for the two-stage stochastic *MPSSP* with complete recourse, whose parameters' uncertainty is represented by a set of scenarios. The first stage variables are the (strategic) 0–1 variables to determine the assignment of the retailers to the facilities. They are structured in special ordered sets, so-called *S3* sets. The second stage variables are the (tactical) continuous variables to determine the product's inventory and backlogging for each time period along the planning horizon under each scenario. We present an approach where the value of the strategic variables do consider all scenarios without being subordinated to any of them. The optimal solution of the second stage variables for each scenario gives a measure of the goodness of the retailers' assignment. A mean cost objective function is considered.

By considering a *splitting variable* mathematical representation of the *DEM* and exploiting the structure of the *S3* sets present in the problem, we specialize the so-called *Branch-and-Fix Coordination (BFC)* algorithmic framework. The specialization makes use of the *Twin Node Family* (*TNF*) concept introduced in [4, 5]. The approach is specifically designed for coordinating the node pruning as well as the selection of the branching node and the branching *S3* set at each *Branch-and-Fix* tree. The *BFC* approach is embedded in a specialization of the so-called *Fix-and-Relax* approach that was introduced in [11] to provide (hopefully) good solutions in deterministic environments. An efficient heuristic scheme, see [13], is used for finding an initial solution to the *MPSSP* as well as for improving feasible solutions at the *TNFs*. The approach, so-called *Fix-and-Relax Coordination (FRC)*, outperforms the plain use of a state-of-the-art optimization system.

The remainder of the paper is organized as follows. Section 2 states the *MPSSP* and introduces the mixed 0–1 *DEM* for the two-stage stochastic version of the problem with complete recourse. Section 3 is devoted to illustrate the *TNF* concept for the *SIP* environment we are dealing with, and to outline the *BFC* specialization approach for the *MPSSP*. The approach is to be executed at the different levels of the *FRC* scheme. Section 4 presents the *FRC* approach. Section 5 reports on the computational results. Section 6 ends the paper with some concluding remarks.

2 Problem description

2.1 Problem statement

Let the *planning horizon* be defined as a set of (consecutive and integer) time periods. Consider a production/distribution network of a single product including a set of *facilities* and a set of *retailers*, see Figure 1. Each facility can be interpreted as a production plant with an associated warehouse. Each retailer needs to be served by (assigned to) a unique facility. The product's demand as well as all costs along the planning horizon are unknown, but it is assumed that the uncertainty can be represented by a set of scenarios. The production and distribution costs are assumed to be stationary while the inventory and the backlogging costs are allowed to be dynamic. Moreover, the production costs are assumed to be linear. Each production plant has a finite, known, possibly time-varying production capacity. We assume that each warehouse has essentially unlimited physical and throughput capacities. That is, its physical capacity is sufficient to be able to store the cumulative excess production of its corresponding production plant, even if this production plant produces to full capacity in each time period. In addition, the throughput capacity is large enough for the warehouse to be able to supply any combination of retailers assigned to it. We assume that the product can only be stored at the facilities, i.e., no storage is allowed at the retailers. This is a realistic assumption for retailers like small supermarkets or restaurants who have a limited storage place available. Backlogging is also allowed at the facilities. The aim is to allocate the retailers to the facilities, so that a cost function is minimized.

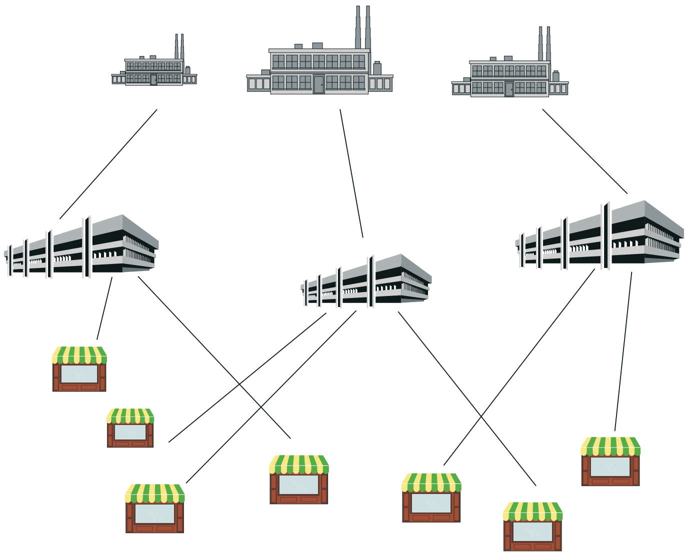

Figure 1: The production/distribution network and the allocations therein

Sets:

$\mathcal{I}$, set of facilities.

$\mathcal{J}$, set of retailers.

$\mathcal{T}$, set of time periods.

Ω, set of scenarios to represent the uncertainty.

Deterministic parameter:

b_{it}, production capacity of facility i in time period t, for $i \in \mathcal{I}, t \in \mathcal{T}$.

Scenario-related and uncertain parameters:

w^{ω}, weight factor assigned to scenario ω, for $\omega \in \Omega$, such that $\sum_{\omega \in \Omega} w^{\omega} = 1$.

d_{jt}^{ω}, product's demand from retailer j in time period t under scenario ω, for $j \in \mathcal{J}$, $t \in \mathcal{T}$, $\omega \in \Omega$.

c_{ij}^{ω}, total assignment cost of retailer j to facility i under scenario ω, consisting of the total production and distribution costs, for $i \in \mathcal{I}$, $j \in \mathcal{J}$, $\omega \in \Omega$.

$h_{it}^{+\omega}$, unit inventory holding cost at facility i in time period t under scenario ω, for $i \in \mathcal{I}$, $t \in \mathcal{T}$, $\omega \in \Omega$.

$h_{it}^{-\omega}$, unit backlogging cost at facility i in time period t under scenario ω, for $i \in \mathcal{I}$, $t \in \mathcal{T}$, $\omega \in \Omega$. We may observe that, for each period, the unit backlogging cost is the same for the non-satisfied demand from any retailer.

Strategic variables: They are 0–1 variables, such that

$$x_{ij} = \begin{cases} 1, & \text{if retailer } j \text{ is assigned to facility } i \\ 0, & \text{otherwise} \end{cases} \qquad \forall i \in \mathcal{I}, j \in \mathcal{J}.$$

Tactical variables: These are continuous variables, such that

$S_{it}^{+\omega}$, product's inventory at facility i in (the end of) time period t under scenario ω, for $i \in \mathcal{I}, t \in \mathcal{T}$, $\omega \in \Omega$.

$S_{it}^{-\omega}$, product's backlogging at facility i in (the end of) time period t under scenario ω, for $i \in \mathcal{I}, t \in \mathcal{T}$, $\omega \in \Omega$.

2.2 Mixed 0–1 Deterministic Equivalent Model (*DEM*)

The following is a *compact* representation of the mixed 0–1 *DEM* for the two-stage stochastic *MPSSP* with complete recourse to minimize the expected cost.

$$\min \sum_{\omega \in \Omega} w^\omega \Big(\sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} c_{ij}^\omega x_{ij} + \sum_{i \in \mathcal{I}} \sum_{t \in \mathcal{T}} h_{it}^{+\omega} S_{it}^{+\omega} + \sum_{i \in \mathcal{I}} \sum_{t \in \mathcal{T}} h_{it}^{-\omega} S_{it}^{-\omega} \Big) \tag{1}$$

subject to

$$\sum_{i \in \mathcal{I}} x_{ij} = 1 \qquad \forall j \in \mathcal{J} \tag{2}$$

$$\sum_{j \in \mathcal{J}} d_{jt}^\omega x_{ij} + S_{it}^{+\omega} + S_{i,t-1}^{-\omega} \leq b_{it} + S_{i,t-1}^{+\omega} + S_{it}^{-\omega} \qquad \forall i \in \mathcal{I}, t \in \mathcal{T}, \omega \in \Omega \tag{3}$$

$$S_{i0}^{+\omega} = S_{i0}^{-\omega} = 0 \qquad \forall i \in \mathcal{I}, \omega \in \Omega \tag{4}$$

$$x_{ij} \in \{0, 1\} \qquad \forall i \in \mathcal{I}, j \in \mathcal{J} \tag{5}$$

$$S_{it}^{+\omega}, S_{it}^{-\omega} \geq 0 \qquad \forall i \in \mathcal{I}, t \in \mathcal{T}, \omega \in \Omega. \tag{6}$$

The objective function consists of the expected assignment, inventory holding and backlogging costs along the planning horizon over the scenarios.

Constraints (2), together with constraints (5), ensure that each retailer is assigned to exactly one facility. The assignment takes into account all scenarios without being subordinated to any of them. These special ordered sets, currently named *S3* sets, were introduced in [7]. Constraints (3) ensure that the production capacity of the facilities is not violated. We can observe that if $\exists i \in \mathcal{I} : S_{it}^{-\omega} > 0$ for $t = |\mathcal{T}|$ for any scenario ω, then the production system cannot satisfy the demand from the retailers and, therefore, it has to be supplied from outside sources. Hence, the model (1)–(6) is always feasible.

We can also observe that the production decisions, say y_{it}^ω, are not explicitly modelled, but we can compute them since the production costs are linear,

$$y_{it}^\omega = \sum_{j \in \mathcal{J}} d_{jt}^\omega x_{ij} + S_{it}^{+\omega} + S_{i,t-1}^{-\omega} - S_{i,t-1}^{+\omega} - S_{it}^{-\omega} \qquad \forall i \in \mathcal{I}, t \in \mathcal{T}.$$

We did not explicitly impose in the model the nonnegativity constraints on these variables, since they are redundant under stationary production and distribution costs, see e.g. [1] for the deterministic case.

We propose an equivalent formulation of the *compact* representation (1)–(6) based on *splitting* the assignment variables. In particular, we replace each variable x_{ij} by x_{ij}^{ω} $\forall w \in \Omega$ and append to the model the so-called *non-anticipativity* constraints

$$x_{ij}^{\omega} - x_{ij}^{\omega+1} = 0 \qquad \forall i \in \mathcal{I}, j \in \mathcal{J}, \omega \in \Omega - \{|\Omega|\} \tag{7}$$

to ensure that the assignments are not subordinated to any of the scenarios.

Let a synthesized version of the *splitting variable* representation of the mixed 0–1 *DEM* (1)–(6) for minimizing the expected (mean) total cost

$$\begin{aligned} Z_{IP} = \min & \textstyle\sum_{\omega\in\Omega} w^{\omega}\Big(c^{\omega}x^{\omega} + h^{\omega}S^{\omega}\Big) & \\ \text{s.t.} & \sum_{i\in\mathcal{I}} x_{ij}^{\omega} = 1 & \forall j \in \mathcal{J}, \omega \in \Omega \\ & D^{\omega}x^{\omega} + BS^{\omega} = b & \forall \omega \in \Omega \\ & x_{ij}^{\omega} - x_{ij}^{\omega+1} = 0 & \forall i \in \mathcal{I}, j \in \mathcal{J}, \omega \in \Omega - \{|\Omega|\} \\ & S_0^{\omega} = 0^{2m} & \forall \omega \in \Omega \\ & x^{\omega} \in \{0,1\}^{mn} & \forall \omega \in \Omega \\ & S^{\omega} \geq 0^{r} & \forall \omega \in \Omega, \end{aligned} \tag{8}$$

where c^{ω} and h^{ω} are the row vectors of the objective function coefficients for the 0–1 and continuous variables, respectively, D^{ω} is the time indexed constraint matrix for the product's demand from the retailers, B is the time indexed constraint matrix $(+1, -1, 0)$ for the product's inventory and backlogging, b is the right-hand-side vector, $x^{\omega} = (x_{ij}^{\omega})_{i\in\mathcal{I},j\in\mathcal{J}}$ gives the mn-vector of the 0–1 variables, S^{ω} gives the r-vector for the continuous variables, where $m = |\mathcal{I}|$, $n = |\mathcal{J}|$ and $r = 2m|\mathcal{T}|$, for $\omega \in \Omega$, and S_0^{ω} the vector for the continuous variables when $t = 0$.

3 Branch-and-Fix Coordination scheme

3.1 Scenario clustering

Notice that the relaxation of the *non-anticipativity* constraints (7) in the model (8) results in a set of $|\Omega|$ independent mixed 0–1 models, where (9) is the model for scenario $\omega \in \Omega$, such that $Z_{IP} = \sum_{\omega\in\Omega} w^{\omega} Z_{IP}^{\omega}$ subject to (7).

$$
\begin{aligned}
Z_{IP}^{\omega} = \min\ & c^{\omega}x^{\omega} + h^{\omega}S^{\omega} \\
\text{s.t. } & \sum_{i\in\mathcal{I}} x_{ij}^{\omega} = 1 \qquad \forall j \in \mathcal{J} \\
& D^{\omega}x^{\omega} + BS^{\omega} = b \\
& S_0^{\omega} = 0^{2m} \\
& x^{\omega} \in \{0,1\}^{mn} \\
& S^{\omega} \geq 0^{r}.
\end{aligned} \tag{9}
$$

It is clear that the relaxation of the constraints (7) is not required for all pairs of scenarios in order to obtain computational efficiency. For reducing the number of subproblems to be solved we reinforce the quality of the relaxation by considering scenario *clustering*. The number of scenario *clusters*, say q, to consider in a given model basically depends on the dimensions of the scenario related model (9), i.e., the parameters $|\mathcal{I}|, |\mathcal{J}|$ and $|\mathcal{T}|$ in our *MPSSP*. Let $\Omega = \cup_{p=1}^{q}\Omega_p$, such that $\Omega_p \cap \Omega_{p'} = \emptyset$, for $p, p' = 1, \ldots, q$ and $p \neq p'$, where Ω_p is the set of scenarios in *cluster* p. Notice that, instead of completely relaxing the non-anticipativity constraints, we impose them within each set Ω_p. The criterion for scenario clustering could be based on the smallest internal deviation of the uncertain parameters, the greatest deviation, etc. This is an open problem and very much instance dependent. The clusters are randomly created in the computational experience that we report in Section 5.

With a slight abuse of the notation, the model to consider for scenario *cluster* $p = 1, \ldots, q$ can be expressed by the *compact* representation

$$
\begin{aligned}
Z_{IP}^{p} = \min\ & \sum_{\omega\in\Omega_p} w^{\omega}\big(c^{\omega}x^{p} + h^{\omega}S^{\omega}\big) \\
\text{s.t. } & \sum_{i\in\mathcal{I}} x_{ij}^{p} = 1 && \forall j \in \mathcal{J} \\
& D^{\omega}x^{p} + BS^{\omega} = b && \forall \omega \in \Omega_p \\
& S_0^{\omega} = 0^{2m} && \forall \omega \in \Omega_p \\
& x^{p} \in \{0,1\}^{mn} \\
& S^{\omega} \geq 0^{r} && \forall \omega \in \Omega_p,
\end{aligned} \tag{10}
$$

where $x^p = (x_{ij}^p)_{i\in\mathcal{I}, j\in\mathcal{J}}$ is the vector of the 0–1 x–variables related to scenario *cluster* p, and the other parameters and variables are as above.

The q problems (10) are linked by the *non-anticipativity* constraints

$$
x_{ij}^{p} - x_{ij}^{p+1} = 0 \qquad \forall i \in \mathcal{I}, j \in \mathcal{J}, p = 1, \ldots, q-1. \tag{11}
$$

Notice that $Z_{IP} = \sum_{p=1}^{q} Z_{IP}^{p}$ subject to (11).

If we were interested in solving just one model (10), we could execute, say, a Branch-and-Bound procedure for ensuring the integrality condition there. We can take benefit from the structure of the constraints (2) on the x-variables. Let (j) denote the *S3* set related to retailer $j \in \mathcal{J}$. The following expression can be used as its *reference row* for branching purposes to solve model (10). Consider

$$C_j^p = \sum_{i \in \mathcal{I}} \hat{c}_{ij}^p x_{ij}^p, \tag{12}$$

where

$$\hat{c}_{ij}^p = \sum_{\omega \in \Omega_p} w^\omega c_{ij}^\omega \qquad \forall i \in \mathcal{I}, j \in \mathcal{J}. \tag{13}$$

We order the members of *S3* (j) in non-decreasing order of the $\hat{c}$–coefficient, and denote by $< i >$ the i-th coefficient after reordering, i.e., $\hat{c}_{<i>j}^p$ is not greater than $\hat{c}_{<i+1>j}^p$, for $i \in \mathcal{I} \setminus \{|\mathcal{I}|\}$. Notice that the potential values of C_j^p (12) in any feasible solution to model (10) are $\hat{c}_{<1>j}^p, \hat{c}_{<2>j}^p, \ldots, \hat{c}_{<m>j}^p$. Therefore, for a given fractional solution for the 0–1 x-variables, say, $\overline{x}_{<i>j}^p \; \forall i \in \mathcal{I}$ there is an index, say, $\overline{\imath} \in \mathcal{I} \setminus \{|\mathcal{I}|\}$ such that

$$\hat{c}_{<\overline{\imath}>j}^p \leq \overline{C}_j^p < \hat{c}_{<\overline{\imath}+1>j}^p, \tag{14}$$

where $\overline{C}_j^p$ is the value of the expression (12) for $x^p = \overline{x}^p$. For each *S3* (j), we propose the following branches

$$\begin{aligned} x_{<i>j}^p &= 0, \qquad i = 1, 2, \ldots, \overline{\imath} && \text{in one branch,} \\ x_{<i>j}^p &= 0, \qquad i = \overline{\imath} + 1, \ldots, m && \text{in the other one.} \end{aligned} \tag{15}$$

This will be the basis for a branching approach when taking into account all models (10) at the same time.

3.2 Twin Node Families

Instead of obtaining independently the optimal solution for each of the models (10), we propose a specialization of the approach so-called *Branch-and-Fix Coordination* (*BFC*) introduced in [4, 5]. It is specially designed to coordinate the selection of the branching node and branching *S3* set for each scenario-related *Branch-and-Fix* (*BF*) tree, such that the relaxed constraints (11) are satisfied when fixing the appropriate variables to either one or zero. The approach also coordinates and reinforces the scenario-related *BF* node pruning, the variable fixing and the objective function bounding of the subproblems attached to the nodes. See in [10, 16, 17, 19, 21, 22, 24], among others, similar decomposition approaches. However, those approaches focus more on using a Lagrangean relaxation of the *non-anticipativity* constraints to obtain good lower bounds, and less on branching and variable fixing. In any case, Lagrangean relaxation schemes can be added on top. See also [26]. (A Benders-Van Slyke-Wets Decomposition approach for two-stage stochastic integer models can be found in [8, 9, 18]. Branch-and-bound approaches for the same type of models can be found in [3, 20]).

For the specialization of the *BFC* approach to solving problem (8), let $\mathcal{R}^p$ denote the *BF* tree associated with scenario *cluster* p, and $\mathcal{A}^p$ the set of active nodes in $\mathcal{R}^p$, $p = 1, \ldots, q$. Any two active nodes, say $a \in \mathcal{A}^p$ and $a' \in \mathcal{A}^{p'}$, $p \neq p'$, are said *twin* nodes if the paths from their *root* node to each of them in their own *BF* trees $\mathcal{R}^p$ and $\mathcal{R}^{p'}$, respectively, have zero-branched (15) on the same x_{ij}–variables (i.e., $x^p_{ij} = x^{p'}_{ij} = 0$, for $i \in \mathcal{I}$, $j \in \mathcal{J}$). Notice that in order to satisfy the *non-anticipativity* constraints (11), the zero-branching and fixing on the *S3* sets must be on the same subsets of the x-variables for the *twin* nodes. A *Twin Node Family* (*TNF*), say, $\mathcal{H}_f$ is a set of nodes, such that any one is a *twin* node to all the other members of the family, for $f \in \mathcal{F}$, where $\mathcal{F}$ is the set of *TNFs*. Notice that $a, a' \in \mathcal{H}_f$, $a \neq a'$, for any family $f \in \mathcal{F}$ implies that $a \in \mathcal{A}^p$ and $a' \in \mathcal{A}^{p'}$, for $p \neq p'$.

We propose the *reference row* (16) for the set *S3* (j), $j \in \mathcal{J}$ to branch in the stochastic setting. The set to consider either takes fractional values or the *non-anticipativity* constraints (11) are not yet satisfied. Consider

$$C_j = \sum_{i \in \mathcal{I}} \sum_{p=1}^{q} \hat{c}^p_{ij} x^p_{ij}, \tag{16}$$

where $\hat{c}^p_{ij}$ is given by (13). So, the multiple branching on the set *S3* (j) can be as in (15) for all members of the given *TNF*, where the index $\bar{\imath}$ is such that

$$\sum_{p=1}^{q} \hat{c}^p_{<\bar{\imath}>j} \leq \overline{C}_j < \sum_{p=1}^{q} \hat{c}^p_{<\bar{\imath}+1>j}, \tag{17}$$

where $\overline{C}_j$ is the value of the expression (16) for the solution $\overline{x}^p$, $\forall p = 1, \ldots, q$.

Let us consider the scenario tree and the *BF* trees shown in Figure 2, where x_{ij} gives the generic notation for the variables x^p_{ij}, $\forall p = 1, \ldots, q$. For illustrative purposes, let the branching order of the *S3* sets be (1), (2). Notice that the first *TNF* to be used is $\mathcal{H}_1$. Based on the linear programming (*LP*) optimal solution of the models (10) attached to the nodes in $\mathcal{H}_1$, let us assume that the branching is as follows: $x_{11} = 0$ in one branch of both *BF* trees and $x_{21} = x_{31} = 0$ in the other branch, so that the *TNFs* $\mathcal{H}_2$ and $\mathcal{H}_3$ are created. As an example, in case that the *TNF* $\mathcal{H}_2$ is selected for branching on the set *S3* (1), the branching on the zero-value for the same variable in both *BF* trees must be performed. So, the families $\mathcal{H}_4$ and $\mathcal{H}_5$ are created, and so forth.

3.3 Algorithmic framework

The following algorithm gives the main steps for solving the original problem (8) by using the scenario *cluster*–related submodels (10), and the *TNF* and *S3* concepts. More details about the current implementation are presented in Section 5.3.

Step 1: Solve the *LP* relaxations of the q models (10). Each model is attached to the *root* node in the trees $\mathcal{R}^p$, $\forall p = 1, \ldots, q$. If the integrality constraints on

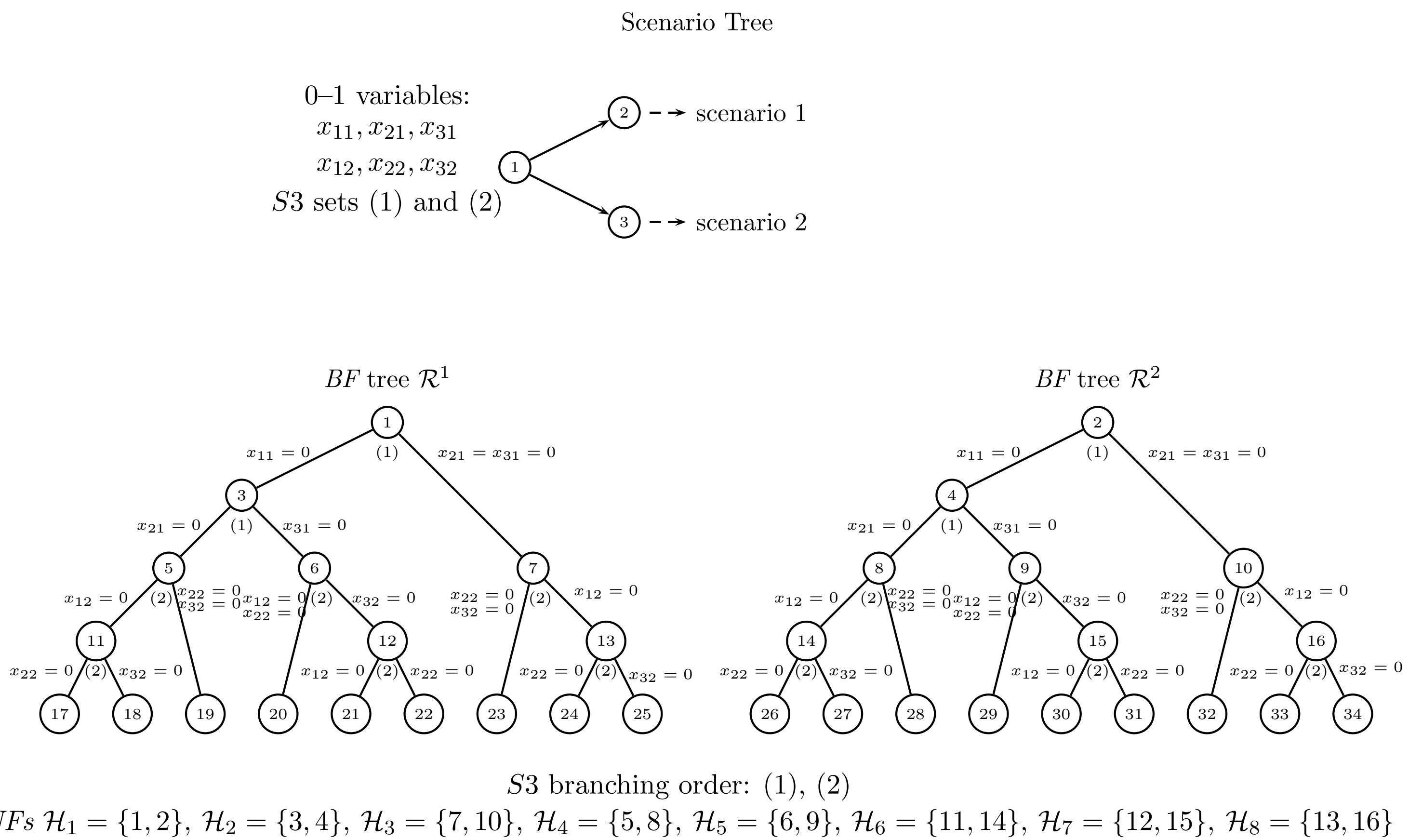

Figure 2: Branch-and-Fix Coordination scheme

the x-variables are satisfied as well as the constraints (11) then stop, the optimal solution to the original mixed 0–1 model has been obtained.

Step 2: The following parameters are saved in a centralized device, so called *Master Device* (*MD*): the values of the variables and the optimal objective function values of the *LP* models attached to the nodes in $\mathcal{A}^p$, $\forall p = 1, \ldots, q$, as well as the appropriate information for branching on the *S3* set in the *TNFs* $\mathcal{H}_f$, $\forall f \in \mathcal{F}$. A decision is made in *MD* for the selection of the *TNF* and the *S3* set to branch. This decision is made available for the execution of each scenario *cluster*–related *BF* phase.

Step 3: Optimization of the *LP* models attached to the newly created nodes from the members of the selected *TNF* by branching on the chosen *S3* set given by (2).

Step 4: In case that the solution that has been obtained in Step 3 has 0–1 values for all the x–variables and it satisfies the constraints (11), a new solution has been found for the original mixed 0–1 model. The *incumbent* solution as well as the sets $\mathcal{A}^p$ at the trees $\mathcal{R}^p$, $\forall p = 1, \ldots, q$, can be updated. In any case, the *TNF* is pruned. If the active node sets are empty, then the optimality of the *incumbent* solution has been proved; otherwise, goto Step 2.

4 Fix-and-Relax Coordination scheme

The *BFC* version considered in the previous section is aimed at obtaining the optimal solution of the original 0–1 problem (8). However, given the combinatorial nature of the problem and the large-scale dimensions of the instances, it is unrealistic to seek for the optimal solution within an affordable computing effort for large instances. Alternatively, we propose the so-called *Fix-and-Relax Coordination* (*FRC*) approach, that aims to obtain ε–quasi optimal solutions for the original problem by selectively exploring some *TNFs* in the *BF* trees.

4.1 Fix-and-Relax methodology

We first consider the *Fix-and-Relax* (*FR*) methodology introduced in [11], and further explored in [14], for obtaining feasible solutions for mixed 0–1 problems. As it is well-known, a Branch-and-Bound scheme to solve, e.g., (9) becomes eventually inefficient (as the number of variables increases) due to the exponential growth in the number of nodes to explore. From a practical point of view, it may even be difficult to find a feasible solution. *FR* is a general purpose methodology that alleviates this difficulty by solving a set of subproblems of smaller complexity than the original problem.

We propose to use the *FR* methodology for the *S3* sets based problem solving. For this purpose let $\mathcal{J}_1, \ldots, \mathcal{J}_k$ be the partitions of the $|\mathcal{J}|$ *S3* sets, such that $\mathcal{J}_g \cap \mathcal{J}_{g'} = \emptyset$,

$\forall g, g' = 1, \ldots, k$ and $g \neq g'$, and $\mathcal{J} = \cup_{g=1}^{k} \mathcal{J}_g$, where k is the given number of partitions to consider. Let us consider the generic problem,

$$\begin{aligned} IP: \quad & \min_{(x,y)\in\mathcal{X}} a_1 x + a_2 y \\ & \text{s.t. } x_{ij} \in \{0,1\}, \qquad \forall i \in \mathcal{I}, j \in \mathcal{J}_g, g = 1, \ldots, k, \end{aligned} \tag{18}$$

where x is the mn–column vector of the 0–1 variables, y is the r–vector of the continuous variables, a_1 and a_2 are the vectors of the related objective function coefficients, and $\mathcal{X}$ is the polytope in $\Re^{mn+r}$ that defines the feasible set.

The *FR* framework requires to solve a sequence of k 0–1 subproblems denoted IP^ℓ, such that each one is attached to the so-called *FR level* $\ell = 1, \ldots, k$. IP^ℓ is defined as follows,

$$\begin{aligned} IP^\ell: \; & \min_{(x,y)\in\mathcal{X}} a_1 x + a_2 y \\ & \text{s.t. } x_{ij} = \overline{x}_{ij} && \forall i \in \mathcal{I}, j \in \mathcal{J}_g, g = 1, \ldots, \ell - 1 \\ & x_{ij} \in \{0,1\} && \forall i \in \mathcal{I}, j \in \mathcal{J}_\ell \\ & x_{ij} \in [0,1] && \forall i \in \mathcal{I}, j \in \mathcal{J}_g, g = \ell + 1, \ldots, k, \end{aligned} \tag{19}$$

where the values $\overline{x}_{ij}$ for $i \in \mathcal{I}, j \in \mathcal{J}_g$, $g = 1, \ldots, \ell - 1$ in *level* $\ell > 1$ are retrieved from the solution to the problems $IP^1, \ldots, IP^{\ell-1}$, respectively.

Since only a reduced subset of the *S3* sets is kept integer at level ℓ it is hoped that IP^ℓ can be solved with relative efficiency. Let us consider the basic *FR* scheme to be used in our approach. For this purpose, let $Z^*(P)$ denote the optimal objective function value (so-called solution value) for a generic problem P in the argument, $\underline{Z}(P)$ be a lower bound on the solution value of problem P, and $Z^2(P)$ be the second best solution value of problem P.

FR: Algorithm Fix-and-Relax

The following framework, based on the *EFRA* algorithm presented in [14], obtains a feasible solution to problem (18) for a given quasi-optimality tolerance, say, ε, such that $(Z^*(IP^k) - \underline{Z}(IP))/\underline{Z}(IP) \leq \varepsilon$.

Input: Partitions $\mathcal{J}_1, \ldots, \mathcal{J}_k$ for a given number of *FR levels* $k \geq 1$, according to the chosen partitioning strategy (see below), and set $\ell := 1$.

Step 1: Solve IP^1. Set $\underline{Z}(IP) := Z^2(IP^1)$.

Step 2: If either $\ell = k$ or all the x–variables in model *IP* take 0–1 values in the optimal solution of model IP^ℓ then stop: The aimed quasi-optimality guarantee of the solution has been achieved. Otherwise, update $\ell := \ell + 1$.

Step 3: Solve IP^ℓ, and update $\underline{Z}(IP) = \min\{\underline{Z}(IP), Z^2(IP^\ell)\}$.
If $(Z^*(IP^\ell) - \underline{Z}(IP))/\underline{Z}(IP) \leq \varepsilon$ then goto Step 2.

Step 4: (Backwards partitioning step). Redefine the partition structure:
$\mathcal{J}_{\ell-1} := \mathcal{J}_{\ell-1} \cup \mathcal{J}_\ell$
$\mathcal{J}_g := \mathcal{J}_{g+1}$, $g = \ell, \ldots, k-1$
$k := k-1$
$\ell := \ell - 1$
If $\ell = 1$ then reset $\varepsilon := \infty$ and goto Step 1. Otherwise, goto Step 3.

Notes:

1: $\underline{Z}(IP)$ is given by the smallest objective function value of the *LP* model attached to any node that has been created so far among all *FR levels* and it has not yet branched on.

2: The *BFC* approach presented in Section 3.3 is used in Step 3 of the *FR* algorithm for solving problem IP^ℓ. The overall approach is named *Fix-and-Relax Coordination (FRC)*.

3: The partition is done according to any of the strategies given in Section 4.2.

4: If all the x–variables in model IP take 0–1 values in any feasible solution of model IP^ℓ then the solution is reinforced by using the heuristic algorithm described in [13] to improve it, see Section 5.2. In this case, the value $Z^*(IP^k)$ is updated, if appropriate.

5: Depending upon the value of the parameter ε, the successive executions of Step 4 of the *FR* algorithm can group the *S3* sets in one single partition, such that the optimal solution of the original problem is sought.

4.2 Partitioning strategies

Let us assume that the strategy for performing the partitions $\mathcal{J}_1, \mathcal{J}_2, \ldots, \mathcal{J}_k$ of the set $\mathcal{J}$ is based upon a given *worth* vector, say, $\beta = (\beta_j \; \forall j \in \mathcal{J})$, such that β_j is to be assigned to *S3* (j). Let us assume a rearrangement of the indices from $\mathcal{J}$ in the non-increasing *worth* order

$$\beta_{<1>} \geq \cdots \geq \beta_{<n>}. \tag{20}$$

Let $n' = \lfloor n/k \rfloor$ be the number of the *S3* sets in each partition but, perhaps, the last one. The first *level* is defined with the partition $\mathcal{J}_{<1>}$ included by the n' most worthy *S3* sets according to the β–worth parameter ordering (20), the second *level* is included by the next n' sets, and so forth.

We have chosen the following partitioning strategies (*ps*), given the conclusions drawn in [14]:

1: *FR Objective Partitioning (FR-OP).* It assigns to each *S3* set a *worth* equal to its average of the objective function coefficients over the scenarios, such that

$$\beta_j = \sum_{i \in \mathcal{I}} \sum_{\omega \in \Omega} w^\omega c_{ij}^\omega.$$

It is a static strategy.

2: *FR Demand Partitioning (FR-DP).* It assigns to each *S3* set a *worth* equal to its weighted product's demand over the scenarios, such that

$$\beta_j = \sum_{t \in \mathcal{T}} \sum_{\omega \in \Omega} w^\omega d_{jt}^\omega.$$

It is a static strategy.

3: *FR Ratio Partitioning (FR-RP).* It assigns to each *S3* set a *worth* consistent with its relative cost. The worth is computed as the ratio between the cost value and the demand required by each *S3* set, such that

$$\beta_j = \sum_{i \in \mathcal{I}} \sum_{\omega \in \Omega} w^\omega c_{ij}^\omega \ / \ \sum_{t \in \mathcal{T}} \sum_{\omega \in \Omega} w^\omega d_{jt}^\omega.$$

It is a static strategy.

4: *FR random Partitioning (FR-rP).* It randomly assigns the *S3* sets to the partitions. For instance, the sets can be arranged in the order they enter in the problem, such that

$$\beta_j = 1/j.$$

It is a static strategy.

5: *FR integrality distance Partitioning (FR-idP).* It assigns to each *S3* set a *worth* equal to the inverse of its average distance from 0.5, such that

$$\beta_j = 1/\sum_{i \in \mathcal{I}} \sum_{\omega \in \Omega} w^\omega |\overline{x}_{ij}^\omega - 0.5 + \eta|,$$

where η is a small positive tolerance and $\overline{x}_{ij}^\omega$ gives the current value of the variable x_{ij}^ω in the procedure. It is a dynamic strategy since the ordering of the *S3* sets depends on the value of the variables at each iteration.

6: *FR weighted Objective Partitioning (FR-wOP).* It assigns to each *S3* set a *worth* equal to its average of the objective function coefficients over the scenarios but weighted by the value $\overline{x}_{ij}^\omega$, such that

$$\beta_j = \sum_{i \in \mathcal{I}} \sum_{\omega \in \Omega} w^\omega c_{ij}^\omega \overline{x}_{ij}^\omega,$$

where $\overline{x}_{ij}^\omega$ is as above. It is a dynamic strategy.

5 Computational experience

5.1 Introduction

In this section we illustrate the performance of a *FRC* approach on random problem instances. Different types of implementations of the *FRC* scheme can be considered within the algorithmic framework presented in the Sections 3 and 4. In Section 5.2 a heuristic to improve the quality of the incumbent integer solution in the *FRC* algorithm is considered. In Section 5.3 the main steps of our implementation of the *BFC* algorithm are described. Finally, in Section 5.4 the computational experiments are reported.

5.2 Heuristic approach

In order to speed up the execution of the *FRC* algorithm, we use the heuristic approach proposed in [13] for the *MPSSP*. This heuristic can be used either for obtaining a feasible solution for the *MPSSP*, or for improving a given one by applying a local search. In particular, we exchange the assignments of two retailers whenever the objective function value of the whole solution improves. We stop when no improving exchange can be found. We take in the root node the initial solution equal to a feasible assignment obtained for the average scenario problem, if the heuristic solution value is improved. Each time the *FRC* algorithm finds a feasible solution, we use the heuristic to improve it.

5.3 BFC implementation

In this section we present the *BFC* implementation for solving up to optimality the problem stated at any *level*, say, ℓ of the *FRC* approach, for $\ell = 1, \dots, k$. For this purpose, we have chosen the *depth first* strategy for the selection of the branching *TNF*.

Notice that only the *S3* sets from $\mathcal{J}_\ell$ are considered to be integer. The criterion for selecting the *S3* set to branch is based on the reduced cost for fixing to zero the variables according to the scheme given in (15), see below.

Once we have chosen the *S3* set, the *reference row* (16) is used to create the two sons of the branching node in each of the *BF* trees. We will say that the two new *TNFs* are *brothers*, and will be indexed with the parameter $\kappa \in \{1, 2\}$.

A *TNF* will be pruned if there is not a guarantee that a better solution than the *incumbent* one can be obtained from the best descendant integer *TNF* (in our implementation, it is based on the *TNF* solution value). Once the *brother TNFs* have been pruned, a *backtracking* is performed to the owner ascendant *TNF*.

For presenting the detailed *BFC* algorithm to solve the model (8) at a given *FR level* ℓ, we introduce the following additional notation where f is used as the branching level index.

$(j)^f$, selected *S3* set in the branching *TNF* $\mathcal{H}_f$.

$LP^p(j)^f(\kappa)$, *LP* relaxation of the scenario *cluster*–related model (10) attached to the κ-th node of the f-th branching for *S3* (j) from the *BF* tree $\mathcal{R}^p$, for $\kappa = 1, 2$, $p = 1, \ldots, q$, $f \in \mathcal{F}$.

$Z^p_{LP}(j)^f(\kappa)$, solution value of the *LP* model $LP^p(j)^f(\kappa)$, for $\kappa = 1, 2$, $p = 1, \ldots, q$, $f \in \mathcal{F}$.

$\underline{Z}_{IP}(j)^f(\kappa)$, lower bound of the solution value of the model attached to the κ-th node of the f-th branching of *S3* (j). It can be computed as $\sum_{p=1}^{q} Z^p_{LP}(j)^f(\kappa)$. By convention, $\underline{Z}_{IP}(0)^0$ for the sum of the *LP* solution values of the root nodes.

$\underline{\alpha}^f_j, \overline{\alpha}^f_j$, the smallest and largest indices of the x-variables that have been branched by fixing them to zero in the f-th branching level of *S3* (j), due to (15). By convention, it is assumed that the indices of the x-variables are ordered according to the non-decreasing criterion shown in the *reference row* (16). Note: It is assumed in the procedure below that $\underline{\alpha}^f_j$ and $\overline{\alpha}^f_j$ inherit the value of the ascendant branching *TNF* $\mathcal{H}_{f-1}$ for branching level f, unless a setting up to a given value is explicitly stated.

$\mathcal{J}^f \subseteq \mathcal{J}_\ell$, set of all the *S3* sets in the f-th branching level that either do not take on integer values, or the *non-anticipativity* constraints are violated.

RC^j_κ, lower bound for the deterioration of the objective value when the κ-th branch of *S3* (j) is selected. It is computed as the sum of the reduced costs associated with the zero-fixing of the variables in the κ-th branch. Note: The variables fixed to zero in the κ-th branch are given by (15).
The criterion to select the *S3* set to branch, $(j)^f$, is as follows,

$$(j)^f \in \arg \max_{(j)\epsilon\mathcal{J}^f} \min_{\kappa=1,2} RC^j_\kappa. \tag{21}$$

Procedure for *FR level* ℓ

Step 0: Set $Z^*(IP^\ell) := \infty$.

Step 1: Solve the *LP* relaxations of the q independent models (10), $\forall p = 1, \ldots, q$, and compute $\underline{Z}_{IP}(0)^0$ for the given *FR level*. If the x-variables from the set $\mathcal{J}_\ell$ take on integer values and the related *non-anticipativity* constraints (11) are satisfied, then an optimal solution to the problem (8) has been found for the given *FR level*, update $Z^*(IP^\ell) := \underline{Z}_{IP}(0)^0$, and stop.

Step 2: Fix $f := 0$, $\underline{\alpha}^f_j := 1$ and $\overline{\alpha}^f_j := |\mathcal{I}|$ for all the *S3* (j) sets.

Step 3: Set $f := f + 1$ and $\mathcal{J}^f$, and select $(j)^f$ as (21).

Let the branching index be denoted by $\hat{\imath}_j^f$, see (17). For each $p = 1, \ldots, q$, one node will have the multiple branching $x_{ij}^p = 0$, $i = \underline{\alpha}_j^f, \ldots, \hat{\imath}_j^f$, and the other one will have the multiple branching $x_{ij}^p = 0$, $i = \hat{\imath}_j^f + 1, \ldots, \overline{\alpha}_j^f$.

Set $\underline{\alpha}_j^f := \underline{\alpha}_j^{f-1}$ and $\overline{\alpha}_j^f := \hat{\imath}_j^f$.

Set $\kappa := 1$.

Step 4: Solve the q independent *LP* models $LP^p(j)^f(\kappa)$, $\forall p = 1, \ldots, q$.

If $\underline{Z}_{IP}(j)^f(\kappa) \geq Z^*(IP^\ell)$ then prune the *TNF* $\mathcal{H}_f$ and goto Step 6.

Step 5: If the x-variables from the set $\mathcal{J}_\ell$ take on integer values and the related *non-anticipativity* constraints (11) are satisfied, then update $Z^*(IP^\ell)$ if appropriate, and goto Step 6. Otherwise, goto Step 3.

Step 6: If the other branch for set $(j)^f$ has already been branched, fixed or pruned then goto Step 7.

Otherwise, set $\underline{\alpha}_j^f := \bar{\imath}_j^f + 1$ and $\overline{\alpha}_j^f := \overline{\alpha}_j^{f-1}$, $\kappa := 2$ and goto Step 4.

Step 7 Update $f := f - 1$. If $f = 0$ then stop, every *TNF* for *FR level* ℓ has been inspected. Otherwise, goto Step 6.

5.4 Numerical results

We report the computational experience obtained while optimizing the model for retailers assignment for a set of instances by using the *FRC* approach presented in the previous sections. For this purpose we have fixed $\varepsilon = 0.03$ and only a backward partitioning step is allowed in the *Fix-and-Relax* algorithm (see Section 4.1). The set of scenarios for the uncertain parameters along the planning horizon has been randomly generated. Table 1 gives the dimensions of the cases.

Our algorithmic approach has been implemented in an experimental C++ code. It uses the optimization engine CPLEX v8.0 for solving the *LP* models at the active nodes in the *BF* trees. The computational experiments were conducted in a PIV with 3.2 Ghz and 1Gb of RAM. The Microsoft Visual C++ compiler v6.0 has been used.

Table 2 gives the dimensions of the scenario– and scenario *cluster*–related deterministic models for $q = 4$. It also gives the dimensions of the deterministic equivalent model, compact representation. The headings are as follows: *nr*, number of restrictions; *n*01, number of 0–1 variables; *nc*, number of continuous variables; and *dens*, constraint matrix density. We can observe the high dimensions of most of the cases.

Table 3 shows the main results of our computational experimentation for solving the original problem, for $q = 4$, $k = 8$ and $ps = 1$. The headings are as follows: Z_{LP}, solution value of the *LP* relaxation of the original problem; $\overline{Z}_{IP}$, value of the incumbent

Table 1: Test bed dimensions

Case	$\|\mathcal{I}\|$	$\|\mathcal{J}\|$	$\|\mathcal{T}\|$	$\|\Omega\|$
C1	3	50	6	50
C2	5	50	6	50
C3	10	50	6	50
C4	5	50	6	100
C5	10	50	6	100
C6	10	100	6	100
C7	10	100	6	200
C8	10	150	6	200
C9	5	100	6	300
C10	10	50	6	300
C11	10	100	6	300
C12	10	150	6	300
C13	5	100	6	400
C14	10	100	6	400
C15	10	150	6	400

solution for the original problem that has been obtained by the plain use of the MIP solver of CPLEX and T^{CPLEX}, related elapsed time (secs) for obtaining it; Z_{LP}^{FRC}, solution value of the *LP* relaxation and Z_{IP}^{FRC}, solution value of the original problem that have been obtained by the *FRC* approach for $q = 4$ scenario *clusters*; *GAP*, optimality gap defined as $(Z_{IP}^{FRC} - Z_{LP}^{FRC})/Z_{LP}^{FRC} \times 100$; *GG*, goodness gap among the CPLEX solution and the *FRC* solution, defined as $(\overline{Z}_{IP} - Z_{IP}^{FRC})/Z_{IP}^{FRC} \times 100$; nf, number of the explored *Twin Node Families*; nn, number of the explored branching nodes for the whole set of *BF* trees; T_{LP}^{FRC} and T_{IP}^{FRC}, the elapsed time (secs) to obtain the *LP* solution and the additional time to obtain the integer solution in the *FRC* approach, respectively; T^{FRC}, total time.

The first conclusion that can be drawn from the results shown in Table 3 is that the optimization engine cannot prove the optimality of the solution within the allowed time limit (7200 secs) in any of the tested cases, even a feasible solution has not been found in two cases. The *GAP* values of the *FRC* approach are usually big. (Notice that the *FRC LP* model is built by also relaxing the *non-anticipativity* constraints (11)). The goodness gap *GG* is greater than 6% in 9 out of the 15 cases and, in some cases, it is very big. Special attention should be given to the *GG* value for the cases C10, C11 and C12. The *FRC* approach gives a good solution for the cases C14 and C15, where CPLEX does not find any. Notice that the *GAP* with respect to Z_{LP} is 18.08% and 7.15% for the cases C14 and C15, respectively.

Tables 4, 5 and 6 show the performance of the *BFC* approach for different sizes of the scenario *clusters* (and, then, different dimensions of model (10)), different values

Table 2: Model dimensions

Case	Scenario model				DEM, compact representation				Scenario cluster model			
	nr	$n01$	nc	dens (%)	nr	$n01$	nc	dens (%)	nr	$n01$	nc	dens (%)
C1	68	150	36	8.87	950	150	1800	2.63	284	150	468	7.28
C2	80	250	60	7.54	1550	250	3000	1.61	440	250	780	4.70
C3	110	500	120	5.48	3050	500	6000	0.82	830	500	1560	2.49
C4	80	250	60	7.54	3050	250	6000	0.85	800	250	1500	2.91
C5	110	500	120	5.48	6050	500	12000	0.43	1550	500	3000	1.50
C6	160	1000	120	4.04	6100	1000	12000	0.79	1600	1000	3000	2.45
C7	160	1000	120	4.04	12100	1000	24000	0.41	3100	1000	6000	1.44
C8	210	1500	120	3.16	12150	1500	24000	0.60	3150	1500	6000	1.96
C9	130	500	60	4.97	9100	500	18000	0.56	2350	500	4500	2.00
C10	110	500	120	5.48	18050	500	36000	0.15	4550	500	9000	0.56
C11	160	1000	120	4.04	18100	1000	36000	0.28	4600	1000	9000	1.02
C12	210	1500	120	3.16	18150	1500	36000	0.41	4650	1500	9000	1.42
C13	130	500	60	4.97	12100	500	24000	0.42	3100	500	6000	1.55
C14	160	1000	120	4.04	24100	1000	48000	0.21	6100	1000	12000	0.79
C15	210	1500	120	3.16	24150	1500	48000	0.31	6150	1500	12000	1.11

for the number of *FR levels*, and different partitioning strategies.

We can observe in Table 4 that the elapsed times for $q = 4$, 6 and 8 are very similar. The elapsed time for $q = 2$ is slightly bigger than for the other values of the parameter, mainly for the cases with higher dimensions. Obviously, the solution value is very similar for all tested values of the q–parameter.

The *BFC* performance that is shown in Table 5 with respect to the *FR levels* shows that the elapsed time for $k = 2$ and 4 in some cases is bigger than the time for $k = 8$ and 16. We can observe that the solution value is very similar for the different values of the k–parameter and, very frequently, it is identical.

The partitioning strategies presented in Section 4.2 seem to be very different. However, we can observe in Table 6 that the selection of the partitioning strategy, ps does not significantly affect any of both results, namely, the solution value and the elapsed time.

Based on the experiments that we have reported, we favor the strategy $q = 4$, $k = 8$ and $ps = 1$. It outperforms the plain use of the optimizer, mainly for the bigger cases that we have tested.

Table 3: Stochastic Solution

Case	Z_{LP}	$\overline{Z}_{IP}$	T^{CPLEX}	Z_{LP}^{FRC}	Z_{IP}^{FRC}	GAP (%)	GG (%)	nf	nn	T_{LP}^{FRC}	T_{IP}^{FRC}	T^{FRC}
C1	23973.84	24569.12	7200	23355.98	24681.03	5.67	−0.45	101	406	0	55	55
C2	17169.17	19084.80	7200	16089.03	19144.86	18.99	−0.31	302	1210	0	95	95
C3	12816.13	17910.67	7200	11961.73	16841.30	40.79	6.35	1878	7514	2	649	651
C4	18409.46	21229.12	7200	17521.19	21234.57	21.19	−0.03	447	1790	1	241	242
C5	15628.26	28660.61	7200	14688.42	26037.25	77.26	10.08	2277	9108	5	2867	2872
C6	30007.01	34241.65	7200	28842.32	33484.07	16.09	2.26	1540	6160	11	5009	5020
C7	29436.66	35787.71	7200	28614.73	32853.19	14.81	8.93	312	1250	45	5074	5119
C8	49327.64	57517.20	7200	48206.63	51870.48	7.60	10.89	312	1248	66	5111	5177
C9	30604.92	31219.27	7200	30034.12	31442.59	4.69	−0.71	951	3806	11	3410	3421
C10	17801.46	155235.42	7200	17072.73	31329.91	83.51	395.49	262	1048	68	5060	5128
C11	24326.05	86644.61	7200	23829.73	27194.73	14.12	218.61	251	1006	73	5060	5133
C12	34715.34	85153.68	7200	34026.23	36779.47	8.09	131.53	166	664	95	5250	5345
C13	39200.46	41094.04	7200	38723.10	40239.34	3.92	2.12	367	1470	61	5052	5113
C14	27467.33	(*)	7200	26704.55	31533.57	18.08	–	108	432	127	5142	5269
C15	38646.11	(*)	7200	37934.43	40646.58	7.15	–	132	528	159	5118	5277

(*) No feasible solution is found in allowed elapsed time: 7200 secs

Table 4: BFC performance. Number of scenario *clusters*, q (for $k = 8$ and $ps = 1$)

Case		$q = 2$	$q = 4$	$q = 6$	$q = 8$
C1	Z_{IP}^{FRC}	24711.24	24681.03	24811.50	24811.50
	nn	272	406	450	450
	T_{IP}^{FRC}	44	55	46	47
C2	Z_{IP}^{FRC}	18890.55	19144.86	19255.19	19255.19
	nn	798	1210	1430	1430
	T_{IP}^{FRC}	101	95	86	85
C3	Z_{IP}^{FRC}	16841.30	16841.30	16841.30	16841.30
	nn	3284	7514	4438	4438
	T_{IP}^{FRC}	560	649	339	341
C4	Z_{IP}^{FRC}	21232.52	21234.57	21609.41	21609.41
	nn	1242	1790	2766	2766
	T_{IP}^{FRC}	371	241	345	346
C5	Z_{IP}^{FRC}	26037.25	26037.25	26037.25	26037.25
	nn	4846	9108	2906	3052
	T_{IP}^{FRC}	5013	2867	5026	5030
C6	Z_{IP}^{FRC}	33484.07	33484.07	33484.07	33484.07
	nn	3210	6160	9184	9184
	T_{IP}^{FRC}	5029	5009	5031	5033
C7	Z_{IP}^{FRC}	32853.20	32853.19	32853.20	32853.20
	nn	848	1250	2062	2058
	T_{IP}^{FRC}	5113	5074	5013	5019
C8	Z_{IP}^{FRC}	51870.48	51870.48	51870.48	51870.48
	nn	454	1248	1088	1088
	T_{IP}^{FRC}	5177	5111	5058	5062
C9	Z_{IP}^{FRC}	31442.58	31442.59	31442.58	31442.58
	nn	1458	3806	2700	2700
	T_{IP}^{FRC}	2184	3410	2414	2427
C10	Z_{IP}^{FRC}	31329.91	31329.91	31329.91	31329.91
	nn	462	1048	1278	1278
	T_{IP}^{FRC}	5097	5060	5034	5034
C11	Z_{IP}^{FRC}	27194.72	27194.73	27194.72	27194.72
	nn	530	1006	1408	1404
	T_{IP}^{FRC}	5119	5060	5032	5020
C12	Z_{IP}^{FRC}	36779.48	36779.47	36779.48	36779.48
	nn	342	664	948	948
	T_{IP}^{FRC}	5423	5250	5146	5144
C13	Z_{IP}^{FRC}	40239.32	40239.34	40239.32	40239.32
	nn	810	1470	2200	2200
	T_{IP}^{FRC}	5097	5052	5037	5042
C14	Z_{IP}^{FRC}	31533.57	31533.57	31533.57	31533.57
	nn	276	432	576	574
	T_{IP}^{FRC}	5291	5142	5087	5098
C15	Z_{IP}^{FRC}	40646.57	40646.58	40646.57	40646.57
	nn	312	528	688	688
	T_{IP}^{FRC}	5324	5118	5073	5073

Table 5: BFC performance. Number of *FR levels*, k (for $q = 4$ and $ps = 1$)

Case		$k = 2$	$k = 4$	$k = 8$	$k = 16$
C1	Z_{IP}^{FRC}	(*)	24813.13	24681.03	24585.54
	nn		1876	406	172
	T_{IP}^{FRC}		52	55	59
C2	Z_{IP}^{FRC}	(*)	19250.89	19144.86	19495.05
	nn		30724	1210	264
	T_{IP}^{FRC}		596	95	74
C3	Z_{IP}^{FRC}	16841.30	16841.30	16841.30	16841.30
	nn	12600	9644	7514	982
	T_{IP}^{FRC}	5028	5009	649	245
C4	Z_{IP}^{FRC}	21710.50	21268.50	21234.57	21356.41
	nn	8278	11514	1790	502
	T_{IP}^{FRC}	5021	5005	241	231
C5	Z_{IP}^{FRC}	26037.25	26037.25	26037.25	26037.25
	nn	2906	8684	9108	1658
	T_{IP}^{FRC}	5026	5009	2867	1263
C6	Z_{IP}^{FRC}	33484.07	33484.07	33484.07	33484.07
	nn	13546	7102	6160	4254
	T_{IP}^{FRC}	4878	5021	5009	4002
C7	Z_{IP}^{FRC}	32853.20	32853.20	32853.19	32853.20
	nn	2052	2304	1250	1404
	T_{IP}^{FRC}	5128	5100	5074	5028
C8	Z_{IP}^{FRC}	51870.48	51870.48	51870.48	51870.48
	nn	1578	1146	1248	1050
	T_{IP}^{FRC}	5347	5083	5111	5068
C9	Z_{IP}^{FRC}	31442.58	31442.58	31442.59	(*)
	nn	4988	6338	3806	
	T_{IP}^{FRC}	5016	5008	3410	
C10	Z_{IP}^{FRC}	26037.25	31329.91	31329.91	31329.91
	nn	2906	1630	1048	834
	T_{IP}^{FRC}	5026	5060	5060	5051
C11	Z_{IP}^{FRC}	27194.72	27194.72	27194.73	27194.72
	nn	1844	1116	1006	846
	T_{IP}^{FRC}	5292	5144	5060	5035
C12	Z_{IP}^{FRC}	36779.48	36779.48	36779.47	36779.48
	nn	648	796	664	424
	T_{IP}^{FRC}	5442	5160	5250	5127
C13	Z_{IP}^{FRC}	40239.32	40239.32	40239.34	40224.01
	nn	2462	1598	1470	956
	T_{IP}^{FRC}	5253	5140	5052	5058
C14	Z_{IP}^{FRC}	31533.57	31533.57	31533.57	31533.57
	nn	588	436	432	328
	T_{IP}^{FRC}	5713	5453	5142	5106
C15	Z_{IP}^{FRC}	40646.57	40646.57	40646.58	40646.57
	nn	566	424	528	360
	T_{IP}^{FRC}	6065	5734	5118	5050

(*) No feasible solution is found in allowed elapsed time: 7200 secs

Table 6: BFC performance. Partitioning strategies, ps (for $q = 4$ and $k = 8$)

Case		$ps = 1$	$ps = 2$	$ps = 3$	$ps = 4$	$ps = 5$	$ps = 6$
C1	Z_{IP}^{FRC}	24681.03	24703.28	24786.46	24678.28	24892.15	24892.15
	nn	406	334	294	254	344	344
	T_{IP}^{FRC}	55	45	35	31	37	38
C2	Z_{IP}^{FRC}	19144.86	18890.42	19133.84	18855.20	19916.79	19916.79
	nn	1210	1008	982	944	742	742
	T_{IP}^{FRC}	95	88	75	97	81	79
C3	Z_{IP}^{FRC}	16841.30	16841.30	16841.30	16618.37	16841.30	16841.30
	nn	7514	6974	4574	3926	4968	4968
	T_{IP}^{FRC}	649	535	470	419	422	423
C4	Z_{IP}^{FRC}	21234.57	20957.61	21557.46	21213.86	21384.01	21384.01
	nn	1790	1620	1196	1540	1924	1924
	T_{IP}^{FRC}	241	259	250	240	261	261
C5	Z_{IP}^{FRC}	26037.25	26037.25	26037.25	25976.68	26037.25	26037.25
	nn	9108	6460	12916	5506	10324	10324
	T_{IP}^{FRC}	2867	2521	4974	2224	3699	3695
C6	Z_{IP}^{FRC}	33484.07	33484.07	33484.07	33484.07	33484.07	33484.07
	nn	6160	6648	6822	7474	7072	7068
	T_{IP}^{FRC}	5009	5013	5008	5018	5008	5009
C7	Z_{IP}^{FRC}	32853.19	32853.19	32853.19	32853.19	32853.19	32853.19
	nn	1250	1602	1538	1224	1280	1286
	T_{IP}^{FRC}	5074	5024	5070	5060	5057	5050
C8	Z_{IP}^{FRC}	51870.48	51870.48	51870.48	51870.48	51870.48	51870.48
	nn	1248	1128	894	1092	1204	1204
	T_{IP}^{FRC}	5111	5092	5123	5082	5104	5103
C9	Z_{IP}^{FRC}	31442.59	31337.53	31337.53	31324.27	31411.75	31411.75
	nn	3806	1464	4044	1628	1674	1674
	T_{IP}^{FRC}	3410	2581	2575	3286	2428	2427
C10	Z_{IP}^{FRC}	31329.91	31329.91	31329.91	31329.91	31329.91	31329.91
	nn	1048	862	892	1182	788	790
	T_{IP}^{FRC}	5060	5107	5058	5041	5044	5046
C11	Z_{IP}^{FRC}	27194.73	27194.73	999999.00	27194.73	27194.73	27194.73
	nn	1006	918	999	1186	1170	1170
	T_{IP}^{FRC}	5060	5088	999	5061	5051	5045
C12	Z_{IP}^{FRC}	36779.47	36779.47	36779.47	36779.47	36779.47	36779.47
	nn	664	616	836	900	944	944
	T_{IP}^{FRC}	5250	5366	5111	5080	5086	5091
C13	Z_{IP}^{FRC}	40239.34	40239.34	40239.34	40239.34	40239.34	40239.34
	nn	1470	1580	1396	1204	1030	1026
	T_{IP}^{FRC}	5052	5034	5086	5081	5064	5064
C14	Z_{IP}^{FRC}	31533.57	31533.57	31533.57	31533.57	31533.57	31533.57
	nn	432	456	452	486	500	500
	T_{IP}^{FRC}	5142	5171	5234	5157	5037	5037
C15	Z_{IP}^{FRC}	40646.58	40646.58	40646.58	40646.58	40646.58	40646.58
	nn	528	390	434	406	390	390
	T_{IP}^{FRC}	5118	5295	5207	5186	5233	5231

6 Conclusions

We have presented a model and an algorithmic approach for the assignment of retailers to facilities under uncertainty. The goodness of each assignment is evaluated through its performance along a planning horizon over the scenarios. The uncertain parameters are the assignment, inventory holding and backlogging costs, and the single product demand from the retailers. We have presented the *Deterministic Equivalent Model* of the two-stage stochastic problem with complete recourse. A *Fix-and-Relax Coordination (FRC)* scheme has been proposed as the solution method. Different partitioning strategies for the *S3* special ordered sets are considered. The approach uses a specialization of the *Branch-and-Fix Coordination (BFC)* scheme to obtain the solution of the subproblem attached to each *Fix-and-Relax* level in the *FRC* scheme. The *BFC* approach takes benefit of the structure of the *S3* sets as well as the *Twin Node Family* concept. The computational experience has been performed with an experimental code. It shows a remarkable reduction in both the solution value and the elapsed time when compared with the plain use of a state-of-the-art optimizer, mainly for cases with high dimensions.

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