Asymptotic analysis of a greedy heuristic for the multi-period single-sourcing problem: the acyclic case*

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August 29, 2003

Abstract

The multi-period single-sourcing problem that we address in this paper can be used as a tactical tool for evaluating logistics network designs in a dynamic environment. In particular, our objective is to find an assignment of customers to facilities, as well as the location, timing and size of production and inventory levels, that minimizes total assignment, production, and inventory costs. We propose a greedy heuristic, and prove that this greedy heuristic is asymptotically optimal in a probabilistic sense for the subclass of problems where the assignment of customers to facilities is allowed to vary over time. In addition, we prove a similar result for the subclass of problems where each customer needs to be assigned to the same facility over the planning horizon, and where the demand for each customer exhibits the same seasonality pattern. We illustrate the behavior of the greedy heuristic, as well as some improvements where the greedy heuristic is used as the starting point of a local interchange procedure, on a set of randomly generated test problems. These results suggest that the greedy heuristic may be asymptotically optimal even for the cases that we were unable to analyze theoretically.

1 Introduction

The tendency to move towards global supply chains causes companies to consider redesigning their logistics network. Most of the quantitative models proposed in the literature for the tactical problem of evaluating (usually with respect to costs) the layout of a distribution network, especially where the assignment of customers to facilities is concerned, assume a static environment. Hence the adequacy of those models is limited to situations where, in particular, the demand pattern is stationary over time. In addition, production and

*This work was supported in part by the National Science Foundation under grant no. DMI-0085682.

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inventory decisions cannot be supported using stationary models. However, the demand for many goods exhibits a seasonal pattern. For instance, the application that motivated our line of research was the case of a beer brewery and distributor, who faces much higher demands in summer, when temperatures are high, than in winter, when temperatures are low. Especially in the presence of production capacities, static models cannot capture issues such as the building of inventory levels in off-peak periods in order to be able to satisfy demands in peak periods, and dynamic models that explicitly take the time dimension of the problem into account need to be used.

In this paper, we will study a multi-period single-sourcing problem (MPSSP) that can be used for evaluating logistics network designs with respect to costs in a dynamic environment. The logistics network consists of a set of facilities, which can be viewed as a plant with an associated warehouse, and a set of customers. Since each warehouse is associated with a particular plant, we assume that any transportation costs between a plant and its warehouse are incorporated in the production costs, and no transportation takes place between warehouses. For a given planning horizon, the customers' demand patterns for a single product are assumed known.

The decisions that need to be made are (i) assignment of customers to facilities, and (ii) location, timing, and size of production and inventory levels. We assume that each plant has a known, finite, and possibly time-varying, production capacity. Moreover, we assume that each associated warehouse has essentially unlimited physical and throughput capacity. In other words, we assume that its physical capacity is sufficient to be able to store the cumulative excess production of its corresponding plant, even if this plant produces to full capacity in each period. In addition, the throughput capacity is large enough for the warehouse to be able to supply any combination of customers assigned to it. Finally, each customer needs to be delivered by (assigned to) a unique facility in each period. The problems in this paper are relevant in short-term planning, where we may assume that initial inventories are given. As is common in standard dynamic lot-sizing problems, we also have a finite planning horizon. To contrast the model studied in this paper to a cyclic variant where initial and ending inventories are variable but equal, which is more relevant for evaluating network designs in longer-term strategic or tactical planning, we call the variant in this paper the *acyclic* case. The cyclic case was studied in Romeijn and Romero Morales [20].

Since this problem is $\mathcal{NP}$-Complete (see Martello and Toth [13] and Romero Morales, Van Nunen and Romeijn [22]), it is unlikely that efficient methods exist that can solve large problem instances to optimality. Therefore, it is appropriate to study heuristic approaches to this problem. We will propose a new family of pseudo-cost functions for the class of greedy heuristics for assignment problems proposed by Martello and Toth [12], in the same spirit as the family of pseudo-cost functions for the Generalized Assignment Problem (GAP) in Romeijn and Romero Morales [15], as well as for the cyclic variant of the MPSSP (see Romeijn and Romero Morales [20] and Romero Morales [21]). We will show particular choices that yield a heuristic that is asymptotically optimal in a probabilistic sense for two large subclasses of problems: (i) problems where the customer assignments are allowed to vary over time; and (ii) problems where the customers need to be assigned to a single facility for the entire planning horizon, and where each customer's demand pattern exhibits the same seasonality pattern. In addition, in the latter case but with a general demand pattern, our numerical results suggest that the heuristic may also be asymptotically optimal.

As mentioned above, most related literature focuses on static models; for example Geoffrion and Graves [9] (whose model also included facility location decisions), Benders et al. [2], and Fleischmann [8]. More recent literature extends some of the traditional models to the dynamic setting. Duran [6] plans the production, bottling, and distribution to agencies of different types of beer, with an emphasis on the production process. Klose [11] analyzes the one-product version of the model proposed by Geoffrion and Graves [9]. Chan, Muriel and Simchi-Levi [3] study a dynamic, but uncapacitated, distribution problem in an operational setting. A multi-period two-echelon multicommodity plant location and inventory planning problem is studied by Hinojosa, Puerto, and Fernández [10], and a greedy heuristic for a three-level (two-echelon) production/distribution system is discussed in Romeijn and Romero Morales [19]. A Lagrangean based heuristic for a multi-level multi-facility and multi-commodity extension of the traditional economic lot-sizing problem is proposed by Wu and Golbasi [23]. Finally, integrated inventory and transportation problems arising in merge-in-transit distribution systems are addressed in Croxton, Gendron and Magnanti [5].

The remainder of the paper is organized as follows. In Section 2 we will formulate the multi-period single-sourcing problem as a mixed-integer linear programming problem, derive some properties of its LP-relaxation, and show the relationship with the GAP through a reformulation of the problem as a pure assignment problem with nonlinear objective function. In Section 3 we will probabilistically analyze the problem. In Section 4 we will introduce a class of greedy heuristics for the problem, and study the asymptotic behavior of a particular element from that class for the two subclasses of problems mentioned above. Numerical experiments will be presented in Section 5, for the greedy heuristic as well as for local exchange procedures for improving a given (partial) solution to the assignment problem. The paper ends in Section 6 with some concluding remarks.

2 The multi-period single-sourcing problem

2.1 A mixed-integer formulation

Let n denote the number of customers, m the number of production and storage facilities, and T the planning horizon. The demand of customer j in period t is given by d_{jt}, while the production capacity at facility i in period t is equal to b_{it}. The costs of assigning customer j to facility i in period t are equal to c_{ijt}, which includes the transportation costs. Note that the transportation costs can be arbitrary functions of demand and distance. Unit production and inventory holding costs at facility i in period t are equal to p_{it} and h_{it}, respectively, and are assumed to be nonnegative. Customer service considerations may necessitate that some or all customers are assigned to the same facility in each period. To incorporate this possibility into the model, we introduce the set $\mathcal{S} \subseteq \{1, \ldots, n\}$ of customers (called static customers) that need to be assigned to the same facility in all periods. We let $\mathcal{D} = \{1, \ldots, n\} \backslash \mathcal{S}$ denote the remaining set of customers (called dynamic customers).

The *multi-period single-sourcing problem (MPSSP)* that we will consider in this paper can now be formulated as follows:

$$\text{minimize} \sum_{t=1}^{T}\sum_{i=1}^{m} p_{it}y_{it} + \sum_{t=1}^{T}\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ijt}x_{ijt} + \sum_{t=1}^{T}\sum_{i=1}^{m} h_{it}I_{it}$$

subject to (P)

$$\begin{aligned}
y_{it} &\leq b_{it} && i=1,\ldots,m;\ t=1,\ldots,T && (1)\\
\sum_{j=1}^{n} d_{jt}x_{ijt} + I_{it} &= y_{it} + I_{i,t-1} && i=1,\ldots,m;\ t=1,\ldots,T && (2)\\
\sum_{i=1}^{m} x_{ijt} &= 1 && j=1,\ldots,n;\ t=1,\ldots,T && (3)\\
x_{ijt} &= x_{ij1} && i=1,\ldots,m;\ j\in\mathcal{S};\ t=2,\ldots,T && (4)\\
I_{i0} &= 0 && i=1,\ldots,m && (5)\\
x_{ijt} &\in \{0,1\} && i=1,\ldots,m;\ j=1,\ldots,n;\ t=1,\ldots,T && (6)\\
y_{it}, I_{it} &\geq 0 && i=1,\ldots,m;\ t=1,\ldots,T
\end{aligned}$$

where x_{ijt} is equal to 1 if customer j is assigned to facility i in period t and 0 otherwise, y_{it} denotes the quantity produced at facility i in period t, and I_{it} denotes the inventory level at facility i at the end of period t. In the remainder of this paper, we will denote the vector of production variables by $y \in \mathbb{R}^{mT}$, and the vector of assignment variables by $x \in \{0,1\}^{nmT}$. In addition, when we consider a subvector of x obtained by fixing one or more of the indices, we will denote this by replacing the remaining indices by $\cdot$. For example, $x_{i\cdot\cdot}$ denotes the subvector of all assignment variables corresponding to facility i.

The objective of (P) is to minimize total production, assignment, and inventory holding costs. The production at facility i in period t is restricted by (1), and constraints (2) are balance equations. Constraints (3) and (6) ensure that each customer is delivered by (assigned to) exactly one facility in each period. Moreover, constraints (4) ensure that each static customer is assigned to the same facility throughout the entire planning horizon. Finally, constraints (5) impose that the inventory at the beginning of the planning horizon is equal to zero in each facility. Due to the nonnegativity of h_{iT}, the inventories at the end of the planning horizon will, without loss of optimality, be equal to zero as well.

Romeijn and Romero Morales [17] provide a probabilistic analysis (an analysis of the behavior of the problem under a stochastic model for the problem parameters) of a variant of the MPSSP where all customers are dynamic and the inventories may exhibit a cyclic pattern for some facilities, i.e., constraints $I_{i0} = 0$ are substituted by $I_{i0} = I_{iT}$ for a subset of the facilities. If this subset is empty, the problem is the same as (P) when all customers are dynamic. They propose a very general probabilistic model for the problem parameters, and derive feasibility conditions and properties of the LP-relaxation under this model that we will use in this paper. A greedy heuristic for the variant of the MPSSP where all facilities exhibit a cyclic inventory pattern was proposed in Romeijn and Romero Morales [20]. It was shown that this heuristic is asymptotically feasible and optimal in a probabilistic sense under a very general probabilistic model on the problem parameters. In this paper, we will analyze a modification of this greedy heuristic for solving (P). The added complexity of (P) is due to the acyclic nature of the problem which makes it *truly* dynamic, whereas the cyclic variant can be reformulated as an essentially static problem. Because of the dynamic nature of the acyclic problem, the analysis developed in Romeijn and Romero Morales [20] does not apply to (P). Still, we will be able to prove asymptotic feasibility and optimality in a probabilistic sense of the greedy heuristic when all customers are dynamic. Due to the difficulty of the acyclic problem in the presence of static customers we are unfortunately not able to prove asymptotic feasibility and optimality of this greedy heuristic in general. However, we will be able to prove asymptotic feasibility and optimality in the presence of static customers for a subclass of problems.

In the following section, we will derive some properties of the LP-relaxation of this problem and its dual. These results will be crucial when proposing a greedy heuristic and analyzing its asymptotic behavior.

2.2 Properties of the LP-relaxation of the MPSSP

The LP-relaxation of (P) can be obtained by replacing the Boolean constraints (6) on x_{ijt} by nonnegativity constraints ($x_{ijt} \geq 0$ for $i = 1, \ldots, m$; $j = 1, \ldots, n$; $t = 1, \ldots, T$). The following lemma derives a bound on the number of split assignments in a basic optimal solution for (LP). This bound will be used in Theorem 4.1 when counting the number of assignments in which the solution given by our greedy heuristic and the one of the LP-relaxation differ. For that, we need to introduce some notation. Let $B_{\mathcal{S}}$ be the set of customers in $\mathcal{S}$ such that $j \in B_{\mathcal{S}}$ means that customer j is split (i.e., customer j is assigned to more than one facility, each satisfying part of his/her demand), and $B_{\mathcal{D}}$ be the set of (customer,period)-pairs such that $(j,t) \in B_{\mathcal{D}}$ means that customer $j \in \mathcal{D}$ is split in period t.

Lemma 2.1 *A basic optimal solution for (LP) satisfies:*

$$|B_{\mathcal{S}}| + |B_{\mathcal{D}}| \leq mT.$$

Proof: Rewrite the problem (LP) with equality constraints and nonnegativity constraints only by introducing slack variables in (1), and eliminating the variables x_{ijt} for $i = 1, \ldots, m$, $j \in \mathcal{S}$ and $t = 2, \ldots, T$, and I_{i0} for $i = 1, \ldots, m$. We then obtain a problem with, in addition to the assignment constraints, $2mT$ equality constraints. Now consider a basic optimal solution to (LP). The number of variables having a nonzero value in this solution is not larger than the number of equality constraints in the reformulated problem. Now note that, for each (i,t), either the production or the associated slack variable in (1) is positive, for a total of mT positive variables. Since there is at least one nonzero assignment variable corresponding to each assignment constraint, and exactly one nonzero assignment variable corresponding to each assignment that is feasible with respect to the integrality constraints of (P), there can be no more than mT assignments that are split. □

After eliminating the variables x_{ijt} (for $i = 1, \ldots, m$, $j \in \mathcal{S}$, and $t = 2, \ldots, T$), using equations (4), and removing equations (3) for $j \in \mathcal{S}$ and $t = 2, \ldots, T$, the dual of (LP) can be formulated as

$$\text{maximize} \sum_{j \in \mathcal{S}} v_j + \sum_{t=1}^{T} \sum_{j \in \mathcal{D}} v_{jt} - \sum_{t=1}^{T} \sum_{i=1}^{m} b_{it} \lambda_{it}$$

subject to (D)

$$\begin{array}{rcll}
v_j & \leq & \sum_{t=1}^{T} (c_{ijt} + \lambda_{it} d_{jt}) & i = 1, \ldots, m;\ j \in \mathcal{S} \\
v_{jt} & \leq & c_{ijt} + \lambda_{it} d_{jt} & i = 1, \ldots, m;\ j \in \mathcal{D};\ t = 1, \ldots, T \\
\lambda_{it} - \mu_{it} & \leq & p_{it} & i = 1, \ldots, m;\ t = 1, \ldots, T \\
\lambda_{i,t+1} - \lambda_{it} & \leq & h_{it} & i = 1, \ldots, m;\ t = 1, \ldots, T-1 \\
-\lambda_{iT} & \leq & h_{iT} & i = 1, \ldots, m \\
\mu_{it} & \geq & 0 & i = 1, \ldots, m;\ t = 1, \ldots, T \\
\lambda_{it} & & \text{free} & i = 1, \ldots, m;\ t = 1, \ldots, T \\
v_j & & \text{free} & j \in \mathcal{S} \\
v_{jt} & & \text{free} & j \in \mathcal{D};\ t = 1, \ldots, T.
\end{array}$$

The dual variables μ_{it} and λ_{it} ($i = 1, \ldots, m$, $t = 1, \ldots, T$) correspond to the production capacity constraints (1) and inventory balance constraints (2), respectively. The dual variable v_j corresponds to the remaining assignment constraint (3) for customer $j \in \mathcal{S}$ and period $t = 1$. Similarly the dual variable v_{jt} corresponds to the assignment constraint (3) for customer $j \in \mathcal{D}$ and period $t = 1, \ldots, T$.

The next result characterizes the split assignments in the optimal solution of (LP). This result gives some intuition when defining the pseudo-cost function used by the greedy heuristic to evaluate assignments of customers to facilities. Moreover, it will also be useful in Section 4 when analyzing the asymptotic feasibility and optimality of our greedy heuristic.

Proposition 2.2 *Suppose that (LP) is feasible and non-degenerate. Let (x^*, y^*, I^*) be a basic optimal solution for (LP) and let (μ^*, λ^*, v^*) be the corresponding optimal solution for (D). Then,*

(i) For each $j \notin B_{\mathcal{S}}$, $x^*_{ijt} = 1$ for $t = 1, \ldots, T$ if and only if

$$\sum_{t=1}^{T} (c_{ijt} + \lambda^*_{it} d_{jt}) = \min_{l=1,\ldots,m} \sum_{t=1}^{T} (c_{ljt} + \lambda^*_{lt} d_{jt})$$

and

$$\sum_{t=1}^{T} (c_{ijt} + \lambda^*_{it} d_{jt}) < \min_{l=1,\ldots,m;\, l \neq i} \sum_{t=1}^{T} (c_{ljt} + \lambda^*_{lt} d_{jt}).$$

(ii) For each $j \in B_{\mathcal{S}}$, there exists an index $i \in \{1, \ldots, m\}$ such that

$$\sum_{t=1}^{T}(c_{ijt} + \lambda_{it}^{*} d_{jt}) = \min_{l=1,\ldots,m;\, l \neq i} \sum_{t=1}^{T}(c_{ljt} + \lambda_{lt}^{*} d_{jt}).$$

(iii) For each $(j,t) \notin B_{\mathcal{D}}$, $x_{ijt}^{*} = 1$ if and only if

$$c_{ijt} + \lambda_{it}^{*} d_{jt} = \min_{l=1,\ldots,m}(c_{ljt} + \lambda_{lt}^{*} d_{jt})$$

and

$$c_{ijt} + \lambda_{it}^{*} d_{jt} < \min_{l=1,\ldots,m;\, l \neq i}(c_{ljt} + \lambda_{lt}^{*} d_{jt}).$$

(iv) For each $(j,t) \in B_{\mathcal{D}}$, there exists an index $i \in \{1, \ldots, m\}$ such that

$$c_{ijt} + \lambda_{it}^{*} d_{jt} = \min_{l=1,\ldots,m;\, l \neq i}(c_{ljt} + \lambda_{lt}^{*} d_{jt}).$$

Proof: Analogous to the proof of Proposition 2.2 in Romeijn and Romero Morales [20]. See also Romeijn and Romero Morales [18] for details. □

In the following section, we will show that (P) can be reformulated as a pure assignment problem with nonlinear cost function and multiple capacity constraints.

2.3 A pure assignment formulation

The original formulation of (P) has assignment variables x_{ijt}, as well as production variables y_{it} and inventory level variables I_{it}. Problem (P) can be reformulated by replacing the production and inventory variables by a nonlinear expression in the assignment variables. The advantage of this is that the problem can be viewed as a pure assignment problem, and that a vector of assignments alone can be used to characterize a solution to the problem. In order to be able to do this we define, for all $i = 1, \ldots, m$, the function $H_i(z)$, $z \in \mathbb{R}_{+}^{nT}$, to be the optimal value to the following problem:

$$\text{minimize } \sum_{t=1}^{T} p_{it} y_{it} + \sum_{t=1}^{T} h_{it} I_{it}$$

subject to

$$\begin{aligned}
y_{it} &\leq b_{it} && t = 1, \ldots, T \\
I_{it} - I_{i,t-1} &= y_{it} - \sum_{j=1}^{n} d_{jt} z_{jt} && t = 1, \ldots, T \\
I_0 &= 0 && \\
y_{it}, I_{it} &\geq 0 && t = 1, \ldots, T.
\end{aligned}$$

Note that this function is finite if and only if $\sum_{t=1}^{\tau} \sum_{j=1}^{n} d_{jt} z_{jt} \leq \sum_{t=1}^{\tau} b_{it}$ for each $\tau = 1, \ldots, T$. We then have the following result:

Theorem 2.3 *Problem (P) can equivalently be formulated as:*

$$\text{minimize} \sum_{t=1}^{T}\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ijt}x_{ijt} + \sum_{i=1}^{m} H_i(x_{i\cdot\cdot})$$

subject to *(P′)*

$$\begin{aligned} \sum_{\tau=1}^{t}\sum_{j=1}^{n} d_{j\tau}x_{ij\tau} &\le \sum_{\tau=1}^{t} b_{i\tau} && i=1,\ldots,m;\ t=1,\ldots,T \qquad (7)\\ \sum_{i=1}^{m} x_{ijt} &= 1 && j=1,\ldots,n;\ t=1,\ldots,T\\ x_{ijt} &= x_{ij1} && i=1,\ldots,m;\ j\in\mathcal{S};\ t=2,\ldots,T\\ x_{ijt} &\in \{0,1\} && i=1,\ldots,m;\ j=1,\ldots,n;\ t=1,\ldots,T. \end{aligned}$$

Proof: Let F be the feasible region of (P). By decomposing (P), we obtain the following equality

$$\begin{aligned} &\min_{(x,y,I)\in F}\left(\sum_{t=1}^{T}\sum_{i=1}^{m} p_{it}y_{it} + \sum_{t=1}^{T}\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ijt}x_{ijt} + \sum_{t=1}^{T}\sum_{i=1}^{m} h_{it}I_{it}\right) =\\ &= \min_{x:\exists(y',I')\,(y',x,I')\in F}\left(\sum_{t=1}^{T}\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ijt}x_{ijt} + \min_{(y,I):(y,x,I)\in F}\left(\sum_{t=1}^{T}\sum_{i=1}^{m} p_{it}y_{it} + \sum_{t=1}^{T}\sum_{i=1}^{m} h_{it}I_{it}\right)\right)\\ &= \min_{x:\exists(y',I')\,(y',x,I')\in F}\left(\sum_{t=1}^{T}\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ijt}x_{ijt} + \sum_{i=1}^{m} H_i(x_{i\cdot\cdot})\right). \end{aligned}$$

It remains to be shown that the feasible region of the decomposed problem is equal to the feasible region of (P′). This follows by, for a given x, choosing

$$\begin{aligned} y_{it} &= b_{it} && i=1,\ldots,m;\ t=1,\ldots,T\\ I_{it} &= y_{it} - \sum_{j=1}^{n} d_{jt}x_{ijt} + I_{i,t-1} && i=1,\ldots,m;\ t=1,\ldots,T. \end{aligned}$$

The nonnegativity constraints on the inventory variables I_{it} are then equivalent to the constraints (7) in (P′). □

It seems intuitively clear that the optimal production and holding costs corresponding to two assignment solutions that nearly coincide should not differ by very much. We formalize this result in the following proposition. This result will be used when showing that the objective value of the solution given by the greedy heuristic and the optimal value of the (LP) are close.

Proposition 2.4 *The function $\sum_{i=1}^{m} H_i$ is a Lipschitz function.*

Proof: See Romeijn and Romero Morales [18]. □

3 A probabilistic analysis of the MPSSP

3.1 General model

Consider the following probabilistic model for the parameters of (P). Note that throughout this paper, random variables will be denoted by capital letters, and their realizations by the corresponding lowercase letters. For each $j = 1, \dots, n$, let (D_j, C_j, Γ_j) be i.i.d. random vectors in $[\underline{D}, \overline{D}]^T \times [\underline{C}, \overline{C}]^{mT} \times \{0, 1\}$ (where $0 < \underline{D} < \overline{D} < \infty$ are lower and upper bounds on the customer demands, and $0 \leq \underline{C} \leq \overline{C} < \infty$ are lower and upper bounds on the assignment costs), where $D_j = (D_{jt})_{t=1,\dots,T}$, $C_j = (C_{ijt})_{i=1,\dots,m;\, t=1,\dots,T}$, and Γ_j has a Bernoulli distribution with parameter $\pi \in [0, 1]$, classifying customer j as static or dynamic as follows:

$$\Gamma_j = \begin{cases} 0 & \text{if } j \in \mathcal{S} \\ 1 & \text{if } j \in \mathcal{D}. \end{cases}$$

We assume that the vectors (D_j, C_j) $(j = 1, \dots, n)$ are i.i.d. according to an absolutely continuous probability distribution for each $j = 1, \dots, n$. We assume that m and T, and therefore also the unit production and holding costs, are fixed. This means that the size of (P) only depends on the number of customers n. We let b_{it} depend linearly on n, i.e., $b_{it} = \beta_{it} n$, for positive constants β_{it}. Note that this dependence of the capacity on the number of customers is necessary to allow the probabilistic model to contain feasible instances as the number of customers grows, and is a common way of modeling capacities in probabilistic models for instances of the GAP (see e.g. Dyer and Frieze [7], Romeijn and Piersma [14], and Romeijn and Romero Morales [16]). For this probabilistic model we have that instances of (LP) are non-degenerate with probability one.

Lemma 3.1 *(LP) is non-degenerate with probability one, under the stochastic model proposed.*

Proof: This follows directly from the fact that the demand parameters are absolutely continuous random variables. □

To be able to talk in a meaningful way about the quality of a heuristic, we wish to constrain ourselves to a class of problem instances that is almost always feasible. On the other hand, we do not want to exclude a significant class of problem instances that are feasible. To analyze the feasibility of (P) under the stochastic model described above, we can use the same methodology as proposed by Romeijn and Piersma [14] for the GAP. As shown by Romeijn and Romero Morales [15] for the GAP, feasibility of the problem instances of (P) is not guaranteed under the above stochastic model, even for the LP-relaxation of (P). Define

$$\Delta = \min_{\lambda \in Q} \left(\lambda^\top \beta - \pi \mathcal{E} \left(\min_{i=1,\dots,m} \sum_{t=1}^{T} \lambda_{it} D_{1t} \right) - (1 - \pi) \sum_{t=1}^{T} \mathcal{E} \left(\min_{i=1,\dots,m} \lambda_{it} D_{1t} \right) \right)$$

where $\mathcal{E}$ denotes expectation, β denotes the vector of capacity parameters β_{it}, and

$$Q = \left\{ \lambda \in \mathbb{R}_+^{mT} \ : \ \sum_{t=1}^{T} \sum_{i=1}^{m} \lambda_{it} = 1;\ \lambda_{i,t+1} \leq \lambda_{it},\ i = 1, \dots, m,\ t = 1, \dots, T-1 \right\}. \quad (8)$$

Theorem 3.2 *As $n \to \infty$, (P) is feasible with probability one if $\Delta > 0$, and infeasible with probability one if $\Delta < 0$.*

Proof: This result can be proven in a similar way as the analogous result for cyclic models (see Romeijn and Romero Morales [20]) and the pure dynamic models (see Romeijn and Romero Morales [17]). The details can also be found in Romero Morales [21]. □

Theorem 3.2 gives an implicit condition for asymptotic feasibility of the instances generated for (P) by this stochastic model. In the following section, explicit conditions are given for particular cases of (P). These conditions will not only be used to prove that the greedy heuristic that we will propose is asymptotically feasible and optimal, but can also be used, for experimental purposes, to generate problem instances which are feasible with probability one. Moreover, these conditions enable us to quantify the tightness of the generated problem instances.

3.2 Static customers

3.2.1 Introduction

In this section, we will analyze the instances for (P) generated by the stochastic model when all customers are static. We have obtained explicit conditions that ensure asymptotic feasibility with probability one for two particular cases. The first case considers the facilities to be identical, i.e., $\beta_{it} = \beta_t$ for each $i = 1, \ldots, m$ and $t = 1, \ldots, T$. (In fact, we will see in Section 3.3 that the condition for this model is equivalent to the condition for the case with identical facilities and only dynamic customers.) In the second model, all customers exhibit the same seasonal demand pattern, i.e., $d_{jt} = \sigma_t d_j$ where σ_t $(t = 1, \ldots, T)$ are nonnegative constants for each $t = 1, \ldots, T$. Under this assumption, (P) can be reformulated as a single-sourcing problem (SSP) with nonlinear Lipschitz objective function (see Proposition 2.4). Since the feasibility is not affected by the cost structure, we can use the result about asymptotic feasibility of the SSP under the stochastic model that Romeijn and Piersma [14] have derived.

3.2.2 Identical facilities

In this section we analyze the special case of (P) with identical facilities, i.e., $\beta_{it} = \beta_t$ for each $i = 1, \ldots, m$ and $t = 1, \ldots, T$.

Theorem 3.3 *Suppose that*

$$m \cdot \sum_{\tau=1}^{t} \beta_\tau > \sum_{\tau=1}^{t} \mathcal{E}(D_{1\tau}) \qquad t = 1, \ldots, T.$$

Then, (P) with static customers and identical facilities is feasible with probability one as n goes to infinity. Moreover, it is infeasible with probability one as n goes to infinity if one of the inequalities is reversed.

Proof: See Romeijn and Romero Morales [18]. □

3.2.3 Seasonal demand pattern

In this section we analyze the special case of (P) where all customers have the same seasonal demand pattern, i.e., $d_{jt} = \sigma_t d_j$ for each $j = 1, \ldots, n$ and $t = 1, \ldots, T$. The equivalent formulation to (P), (P′), shows us that (P) with static customers and seasonal demand pattern can be rewritten as a SSP with Lipschitz objective function.

Proposition 3.4 *Problem (P) with static customers and seasonal demand pattern can be reformulated as:*

$$\text{minimize} \sum_{i=1}^{m}\sum_{j=1}^{n}\left(\sum_{t=1}^{T} c_{ijt}\right) x_{ij1} + \sum_{i=1}^{m} H_i\left(x_{i\cdot 1}, \ldots, x_{i\cdot 1}\right)$$

subject to

$$\begin{aligned} \sum_{j=1}^{n} d_j x_{ij1} &\leq \min_{t=1,\ldots,T}\left(\frac{\sum_{\tau=1}^{t} b_{i\tau}}{\sum_{\tau=1}^{t}\sigma_\tau}\right) && i = 1, \ldots, m \\ \sum_{i=1}^{m} x_{ij1} &= 1 && j = 1, \ldots, n \\ x_{ij1} &\in \{0, 1\} && i = 1, \ldots, m;\ j = 1, \ldots, n \end{aligned}$$

where $\sum_{i=1}^{m} H_i$ *is a Lipschitz function.*

Proof: The result follows by combining Theorem 2.3 and Proposition 2.4. □

Then, by using the result on asymptotic feasibility of the SSP from Romeijn and Piersma [14], the following assumption ensures asymptotic feasibility with probability one.

Assumption 3.5 *Assume that*

$$\sum_{i=1}^{m} \min_{t=1,\ldots,T}\left(\frac{\sum_{\tau=1}^{t} \beta_{i\tau}}{\sum_{\tau=1}^{t}\sigma_\tau}\right) > \mathcal{E}(D_1).$$

In Section 4 we will show that, under this assumption, our heuristic provides a feasible solution to the problem with probability one. The tightness of this condition is illustrated by noting that, in addition, asymptotic *infeasibility* of problem instances is guaranteed with probability one if the inequality in the assumption is reversed.

3.3 Dynamic customers

Romeijn and Romero Morales [17] have derived explicit conditions to ensure asymptotic feasibility of the instances with probability one when all customers are dynamic. In particular, they have shown that the following assumption ensures asymptotic feasibility for the subclass of problems where the customers are dynamic:

Assumption 3.6 *Assume that*

$$\sum_{\tau=1}^{t} \mathcal{E}(D_{1\tau}) < \sum_{\tau=1}^{t}\sum_{i=1}^{m} \beta_{i\tau} \quad t = 1, \ldots, T.$$

We will show in Section 4 that our heuristic provides a feasible solution to the problem with probability one under the same condition. The tightness of this condition is illustrated by noting that asymptotic *infeasibility* of problem instances is guaranteed with probability one if at least one of the inequalities in the assumption is reversed.

Note that the condition in Theorem 3.3 is equivalent to Assumption 3.6, and thus the condition for asymptotic feasibility of (P) for static customers and identical facilities is the same as the asymptotic feasibility condition for dynamic customers.

4 An asymptotically optimal greedy heuristic

4.1 A class of greedy heuristics

The class of greedy heuristics we propose in this section is conceptually similar to the one proposed by Martello and Toth [12] for the GAP. (Recall from Section 2.3 that (P) can be formulated as a pure assignment problem.) The idea is that each possible assignment is evaluated by a pseudo-cost function $f(i, j, t)$. For each assignment to be made, the difference between the second smallest and the smallest values of $f(i, j, t)$ (called the *desirability* of making the cheapest assignment with respect to the pseudo-cost) is computed, and assignments are made in decreasing order of this difference. Along the way, the remaining capacities of the facilities, and consequently the values of the desirabilities, are updated to maintain feasibility.

Greedy heuristic

Step 0. Set $L = \{1, \ldots, n\} \times \{1, \ldots, T\}$, $B_{it} = \sum_{\tau=1}^{t} b_{i\tau}$ for each $i = 1, \ldots, m$ and $t = 1, \ldots, T$, and $x^{\mathrm{G}} = 0$.

Step 1. For all $(j, t) \in L$, let

$$\begin{aligned}
\mathcal{F}_{jt} &= \{i : \sum_{\tau=1}^{\overline{\tau}} d_{j\tau} \le B_{i\overline{\tau}},\ \overline{\tau} = 1, \ldots, T\} \text{ for } (j, t) \in L \cap (\mathcal{S} \times \{1, \ldots, T\}) \\
\mathcal{F}_{jt} &= \{i : d_{jt} \le B_{it}\} \text{ for } (j, t) \in L \cap (\mathcal{D} \times \{1, \ldots, T\}).
\end{aligned}$$

If $\mathcal{F}_{jt} = \emptyset$ for some $(j,t) \in L$: let $L = L \backslash \{(j,t)\}$ and repeat Step 1. Otherwise, let

$$\begin{array}{rcll} i_{jt} & \in & \arg\min_{i \in \mathcal{F}_{jt}} f(i,j,t) & \text{for } (j,t) \in L \\ \rho_{jt} & = & \min_{\substack{s \in \mathcal{F}_{jt} \\ s \neq i_{jt}}} f(s,j,t) - f(i_{jt},j,t) & \text{for } (j,t) \in L. \end{array}$$

Step 2. Let $(\hat{\jmath},\hat{t}) \in \arg\max_{(j,t)\in L} \rho_{jt}$. If $\hat{\jmath} \in \mathcal{D}$, set

$$\begin{array}{rcl} x^{\mathrm{G}}_{i_{\hat{\jmath}\hat{t}},\hat{\jmath}\hat{t}} & = & 1 \\ L & = & L \setminus \{(\hat{\jmath},\hat{t})\}, \end{array}$$

and if $\hat{\jmath} \in \mathcal{S}$, set

$$\begin{array}{rcll} x^{\mathrm{G}}_{i_{\hat{\jmath}\hat{t}},\hat{\jmath}t} & = & 1 & \text{for } t = 1,\ldots,T \\ L & = & L \setminus \{(\hat{\jmath},t) : t = 1,\ldots,T\} & \end{array}$$

and update the aggregate capacities B_{it}.

Step 3. If $L = \emptyset$: STOP, x^{G} is a (partial) solution to (P′). Otherwise, go to Step 1.

In Step 1 of the heuristic, we first determine the set of facilities to which each remaining (customer,period)-pair can feasibly be assigned given the remaining capacities. Note that a facility can feasibly supply a (customer,period)-pair for a static customer if it can supply the customer for *all* periods. Next, the desirability of making each feasible assignment is determined by finding the differences between the minimum and second minimum pseudo-cost for each remaining (customer,period)-pair. In Step 2 of the heuristic, the most desirable assignment is determined and realized, and the remaining capacities for each of the facilities in each period are determined. Note that this step explicitly ensures that customers in $\mathcal{S}$ are assigned to the same facility in each period. Since, for static customers $j \in \mathcal{S}$, $\mathcal{F}_{jt}$ (in Step 1) is independent of t, the demand of that customer in all periods is taken into account when determining the most desirable facility for that customer.

The output of the heuristic is a vector of assignments x^{G}, which is, in general, a *partial* solution, i.e., a solution in which not all pairs (j,t) are assigned to a facility, of the reformulated problem (P′).

In the following section we show that, for a particular choice of the pseudo-cost function, this greedy heuristic is asymptotically feasible and optimal in a probabilistic sense when all customers are dynamic. We will prove the same for the static case with seasonal demand pattern.

4.2 Asymptotic optimality of a greedy heuristic

4.2.1 A particular pseudo-cost function

Romeijn and Romero Morales [20] have shown asymptotic feasibility and optimality in the probabilistic sense for the cyclic model when the pseudo-cost function is defined as

$$f(i,j,t) = \begin{cases} \sum_{\tau=1}^{T}(c_{ij\tau} + \lambda^*_{i\tau} d_{j\tau}) & \text{if } j \in \mathcal{S} \\ c_{ijt} + \lambda^*_{it} d_{jt} & \text{if } j \in \mathcal{D} \end{cases}$$

where $\lambda^* \geq 0$ represents the vector of optimal dual multipliers of the inventory balance constraints (2) of the LP-relaxation of the cyclic model. The reformulation as a pure assignment problem of this model yields an SSP with nonlinear objective function where the capacity of each facility is defined as the aggregate capacity over time. Using this fact, it was possible to show asymptotic feasibility and optimality with probability one of this greedy heuristic. The general idea of the proof is to show that (i) the number of differences between the assignments in the optimal solution of the LP-relaxation of the cyclic model and the partial solution given by the greedy heuristic is bounded from above by a constant independent of n; and (ii) the feasibility condition in the stochastic model that guarantees asymptotic feasibility with probability one also ensures that the greedy heuristic will, in an asymptotic sense, find a feasible solution with probability one. These two properties imply that the normalized optimal values of the cyclic model and its LP-relaxation tend, with probability one, to the same constant when n goes to ∞, thus implying that the greedy heuristic is asymptotically optimal with probability one.

For *acyclic* problems, we use essentially the same pseudo-cost function, replacing $\mu^* \geq 0$ by the vector of optimal dual multipliers of the capacity constraints of LP-relaxation of the *acyclic* model. We will apply a similar proof technique in this paper to show asymptotic optimality of the greedy heuristic for the two subclasses of the acyclic MPSSP mentioned earlier. The added difficulty we have to overcome is the truly dynamic nature of the assignments. Throughout this section, let x^{LP} denote the optimal assignments of the LP-relaxation of (P) (and thus, using the same argument as in Theorem 2.3, the optimal solution for the relaxation of (P′)), z^{LP} be its objective value, x^{G} denote the (partial) solution for (P′) given by the greedy heuristic, and z^{G} be its objective value. Let N be the set of assignments which do not coincide in x^{G} and in x^{LP}. We will first show that the number of differences between the assignments in x^{LP} and x^{G} can be bounded from above by a constant independent of n. Recall that $0 < \underline{D} < \overline{D} < \infty$ are lower and upper bounds on the demands, respectively.

Theorem 4.1 *There exists some constant R, independent of n, so that $|N| \leq R$ for all instances of (LP) that are feasible and non-degenerate.*

Proof: See the Appendix. □

Even if the partial solution constructed by the greedy heuristic is very close to the partial solution consisting of all feasible assignments in the optimal solution to the LP-relaxation, the greedy heuristic could fail to assign some of the static customers or some of the (customer,period)-pairs for dynamic customers. In the following two subsections we will prove that, under the appropriate feasibility condition on the stochastic model, this happens only with probability zero as $n \to \infty$ for problems with static customers and seasonal demands. Note that, for this case, the MPSSP is essentially a static SSP, as shown in Section 3.2.3. Unfortunately, we have not been able to derive a similar result for the problem with static customers and general demands. The complication in this truly dynamic problem is that customers have to be assigned to a single facility throughout the planning horizon, thereby prohibiting any kind of sequential analysis of the problem. This is in contrast to the case of dynamic customers, where we propose a local exchange heuristic to be applied following the greedy heuristic. We will show that the resulting combined heuristic is asymptotically feasible and optimal as $n \to \infty$.

4.2.2 The static case with seasonal demand pattern

We will use the following lemma in the proof of Theorem 4.3.

Lemma 4.2 *Under Assumption 3.5,*

$$\sum_{i=1}^{m} \min_{t=1,\ldots,T} \left(\frac{\sum_{\tau=1}^{t} \beta_{i\tau}}{\sum_{\tau=1}^{t} \sigma_\tau} \right) - \frac{1}{n} \sum_{i=1}^{m} \sum_{j=1}^{n} D_j X_{ij}^{\mathrm{LP}} > 0$$

with probability one when n *goes to infinity, where* X_{ij}^{LP} *denotes an optimal solution to the LP-relaxation of the problem.*

Proof: Note that

$$\begin{aligned}
&\sum_{i=1}^{m} \min_{t=1,\ldots,T} \left(\frac{\sum_{\tau=1}^{t} \beta_{i\tau}}{\sum_{\tau=1}^{t} \sigma_\tau} \right) - \frac{1}{n} \sum_{i=1}^{m} \sum_{j=1}^{n} D_j X_{ij}^{\mathrm{LP}} \\
&= \sum_{i=1}^{m} \min_{t=1,\ldots,T} \left(\frac{\sum_{\tau=1}^{t} \beta_{i\tau}}{\sum_{\tau=1}^{t} \sigma_\tau} \right) - \frac{1}{n} \sum_{j=1}^{n} D_j \left(\sum_{i=1}^{m} X_{ij}^{\mathrm{LP}} \right) \\
&= \sum_{i=1}^{m} \min_{t=1,\ldots,T} \left(\frac{\sum_{\tau=1}^{t} \beta_{i\tau}}{\sum_{\tau=1}^{t} \sigma_\tau} \right) - \frac{1}{n} \sum_{j=1}^{n} D_j \\
&\to \sum_{i=1}^{m} \min_{t=1,\ldots,T} \left(\frac{\sum_{\tau=1}^{t} \beta_{i\tau}}{\sum_{\tau=1}^{t} \sigma_\tau} \right) - \mathcal{E}(D_1)
\end{aligned}$$

with probability one as $n \to \infty$. Thus, the desired result follows by using Assumption 3.5. □

Theorem 4.3 *Under Assumption 3.5, the greedy heuristic proposed in Section 4.1 is asymptotically* feasible *with probability one.*

Proof: (LP) is non-degenerate with probability one (see Lemma 3.1) and feasible with probability one when $n \to \infty$ by using Assumption 3.5. Thus, from Theorem 4.1, we know that the number of assignments that differ between the optimal solution of the relaxation of (P′) and the solution given by the greedy heuristic is bounded from above by a constant independent of n. Moreover, Lemma 4.2 ensures us that the remaining capacity in the optimal solution for the relaxation of (P′) grows linearly with n. Thus, when n grows to infinity, the (partial) solution found by the greedy heuristic is a feasible solution to (P′). □

We are now ready to prove asymptotic optimality of the greedy heuristic.

Theorem 4.4 *Under Assumption 3.5, the greedy heuristic proposed in Section 4.1 is asymptotically* optimal *with probability one.*

Proof: From Theorem 4.3 we know that the greedy heuristic is asymptotically feasible with probability one. Denoting the objective value of the greedy solution of an instance with n customers by Z_n^{G} and the optimal solution value to the corresponding LP-relaxation by Z_n^{G}, it thus suffices to show that $|\frac{1}{n}Z_n^{\mathrm{LP}} - \frac{1}{n}Z_n^{\mathrm{G}}| \to 0$ with probability one as $n \to \infty$. This follows directly from Theorem 4.1 and Proposition 2.4. □

4.2.3 The dynamic case

Let $\mathcal{U}$ be the set of (customer, period)-pairs which could not be assigned by the greedy heuristic, i.e.,

$$\mathcal{U} = \left\{(j,t) : x^{\mathrm{G}}_{\cdot jt} = 0\right\}.$$

We describe a local exchange procedure to try to assign the pairs in $\mathcal{U}$. After fixing the assignments from the greedy heuristic and calculating the corresponding optimal production levels y^{G}, the remaining aggregate capacities are equal to

$$\sum_{t=1}^{\tau} b_{it} - \sum_{t=1}^{\tau} y^{\mathrm{G}}_{it}$$

for each $i = 1, \ldots, m$ and $\tau = 1, \ldots, T$. The elements of the set $\mathcal{U}$ will be considered in increasing order of the period to which they belong. Suppose that we are in a particular period τ and pair $(\hat{\jmath}, \tau)$ is the next element of $\mathcal{U}$ under consideration. If there exists a facility i for which the remaining capacity in periods $t = \tau, \ldots, T$ is at least $d_{\hat{\jmath}\tau}$, i.e.,

$$\sum_{t=1}^{\overline{\tau}} b_{it} - \sum_{t=1}^{\overline{\tau}} \sum_{j=1}^{n} d_{jt} x^{\mathrm{G}}_{ijt} > d_{\hat{\jmath}\tau} \qquad \text{for } \overline{\tau} = \tau, \ldots, T$$

then we can assign $(\hat{\jmath}, \tau)$ to facility i. Otherwise, if there exists a facility i such that the remaining capacity in period τ is at least $d_{\hat{\jmath}\tau}$, i.e.,

$$\sum_{t=1}^{\tau} b_{it} - \sum_{t=1}^{\tau} \sum_{j=1}^{n} d_{jt} x^{\mathrm{G}}_{ijt} > d_{\hat{\jmath}\tau} \tag{9}$$

then we can still assign $(\hat{\jmath},\tau)$ to facility i by removing some assignments in later periods, and adding those to the set $\mathcal{U}$. We will then proceed with the next element of $\mathcal{U}$. If there is no facility for which inequality (9) holds, then the local exchange procedure is not able to assign $(\hat{\jmath},\tau)$. We will show that this occurs with probability zero as $n \to \infty$.

Local exchange procedure for feasibility

Step 0. Let x^{G} be the current partial solution for (P′). Set $\mathcal{U} = \{(j,t) : x^{\mathrm{G}}_{\cdot jt} = 0\}$, $\mathcal{I} = \emptyset$, and $\tau = 1$.

Step 1. If $\{j : (j,\tau) \in \mathcal{U}\} = \emptyset$, go to Step 4. Otherwise, choose $\hat{\jmath} = \arg\max_{j:(j,\tau)\in\mathcal{U}} d_{j\tau}$ (where ties are broken arbitrarily).

Step 2. If there exists some facility $\hat{\imath}$ such that inequality (9) holds for every $\overline{\tau} = \tau, \ldots, T$ then set

$$\begin{aligned} x^{\mathrm{G}}_{\hat{\imath},\hat{\jmath},\tau} &= 1 \\ \mathcal{U} &= \mathcal{U} \setminus \{(\hat{\jmath},\tau)\} \end{aligned}$$

and go to Step 1. Otherwise, if $\tau = T$, set

$$\begin{aligned} \mathcal{I} &= \mathcal{I} \cup \{(\hat{\jmath},\tau)\} \\ \mathcal{U} &= \mathcal{U} \setminus \{(\hat{\jmath},\tau)\} \end{aligned}$$

and go to Step 1.

Step 3. If there exists some facility $\hat{\imath}$ such that inequality (9) holds for $\overline{\tau} = \tau$ then find a collection of pairs $(j_1,t_1),\ldots,(j_s,t_s)$ so that $x^{\mathrm{G}}_{\hat{\imath}j_kt_k} = 1$ where $t_k > \tau$ for each $k = 1,\ldots,s$, such that reversing the assignments in this collection makes the assignment of $(\hat{\jmath},\tau)$ to $\hat{\imath}$ feasible. Then set

$$\begin{aligned} x^{\mathrm{G}}_{\hat{\imath},\hat{\jmath},\tau} &= 1 \\ x^{\mathrm{G}}_{\hat{\imath},j_k,t_k} &= 0 \qquad \text{for } k = 1\ldots,s \\ \mathcal{U} &= \left(\mathcal{U} \cup \bigcup_{k=1}^{s}\{(j_k,t_k)\}\right) \setminus \{(\hat{\jmath},\tau)\} \end{aligned}$$

and go to Step 1. If such a facility does not exist, set

$$\begin{aligned} \mathcal{I} &= \mathcal{I} \cup \{(\hat{\jmath},\tau)\} \\ \mathcal{U} &= \mathcal{U} \setminus \{(\hat{\jmath},\tau)\} \end{aligned}$$

and go to Step 1.

Step 4. If $\tau = T$, STOP: x^{G} is a feasible solution to (P′) if $\mathcal{I} = \emptyset$, and otherwise it is a partial solution. If $\tau < T$, increment τ by one and go to Step 1.

In Theorem 4.5, we show that the greedy heuristic given in Section 4.1 followed by the local exchange procedure for feasibility is asymptotically feasible with probability one. This proof is based on the result of Theorem 4.1, which says that x^{LP} and x^{G} coincide for almost all assignments in x^{LP}.

Theorem 4.5 *Under Assumption 3.6, the greedy heuristic given in Section 4.1 combined with the local exchange procedure for feasibility is asymptotically* feasible *with probability one.*

Proof: See the Appendix. □

In Theorem 4.6, we show that the greedy heuristic combined with the local exchange for feasibility is asymptotically optimal with probability one. The proof is similar to the proof of Theorem 4.4.

Theorem 4.6 *Under Assumption 3.6, the greedy heuristic given in Section 4.1 followed by the local exchange procedure for feasibility is asymptotically* optimal *with probability one.*

Proof: Similar to the proof of Theorem 4.4. □

5 Computational results

In this section we will illustrate the behavior of the greedy heuristic described in Section 4 on a set of randomly generated test problems. We have considered a collection of classes of instances for the acyclic MPSSP including the two special cases for which we were able to prove asymptotic optimality in the probabilistic sense, namely instances where all customers are dynamic and for those ones where all the customers are static and exhibiting the same seasonal demand pattern. The variety is based on the type of facility, the type of demand pattern and the ratio between dynamic and static customers. More precisely, we have considered:

- identical facilities and a general demand pattern;
- non-identical facilities and a general demand pattern;
- non-identical facilities and a seasonal demand pattern;

and for each of those cases, we have generated instances for:

- the purely dynamic case, i.e., $\mathcal{D} = \{1, \ldots, n\}$ and $\mathcal{S} = \emptyset$;
- the purely static case, i.e., $\mathcal{S} = \{1, \ldots, n\}$ and $\mathcal{D} = \emptyset$;
- and a mixed case, where the probabilities that a customer is static or dynamic are both equal to $\frac{1}{2}$, i.e., $\mathcal{E}(|\mathcal{D}|) = \mathcal{E}(|\mathcal{S}|) = \frac{1}{2}n$.

Our base scenarios have $m = 5$ facilities and $T = 6$ periods. To study the effect of increasing the number of facilities or periods on the behavior of the heuristic, we have considered the case of $m = 10$ facilities and $T = 6$ periods, as well as the case of $m = 5$ facilities and $T = 12$ periods. For each problem instance, we have generated a set of customers and a set of facilities uniformly in the square $[0, 10]^2$.

When all customers exhibit the same seasonal demand pattern, for each customer we have generated D_j from the uniform distribution on $[5, 25]$, and then $D_{jt} = \sigma_t D_j$. For the more general case, we have generated for each customer a random demand D_{jt} in period t from the uniform distribution on $[5\sigma_t, 25\sigma_t]$. To simulate peak demand periods preceded and followed by intermediate and off-peak periods, we have chosen the vector of seasonal factors $\sigma = (\sigma_1, \ldots, \sigma_T)^\top$ as shown in Figure 1. The costs C_{ijt} are assumed to be proportional to demand and distance, i.e., $C_{ijt} = D_{jt} \cdot \text{dist}_{ij}$, where dist_{ij} denotes the Euclidean distance between facility i and customer j. Finally, we have generated inventory holding costs H_{it} uniformly from $[10, 30]$, and assumed (without loss of generality) that the unit production costs are equal to zero.

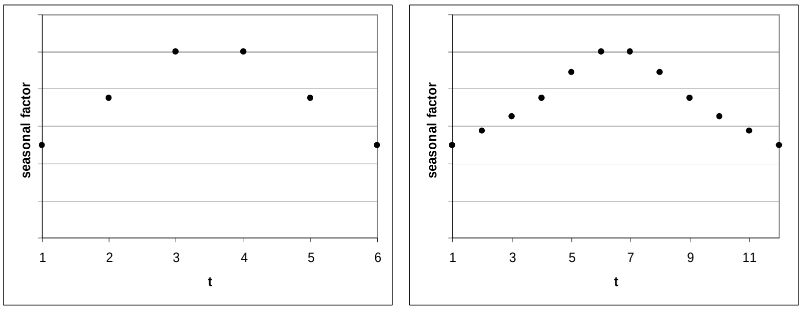

Figure 1: Seasonal factors for $T = 6$ and $T = 12$.

We have chosen the capacities to be of the form $b_{it} = \omega_i \cdot \beta \cdot n$, where

$$\beta = \delta \cdot 15 \cdot \max_{t=1,\ldots,T} \left(\frac{1}{t} \sum_{\tau=1}^{t} \sigma_\tau \right).$$

This choice of capacities is motivated by the feasibility condition on the probabilistic model for the problem instances mentioned in Section 3. To ensure asymptotic feasibility with probability one for the two classes earlier mentioned, we need to choose $\sum_{i=1}^{m} \omega_i = 1$ and $\delta > 1$. To account for the asymptotic nature of this feasibility guarantee, we have set $\delta = 1.1$ to obtain a reasonable number of feasible instances for finite n, while keeping the capacities low in order to obtain hard problem instances. Clearly, the case of identical facilities corresponds to $\omega = (\frac{1}{m}, \ldots, \frac{1}{m})^\top$. To illustrate the case of non-identical facilities, we have chosen $\omega = (\frac{1}{10}, \frac{1}{10}, \frac{1}{5}, \frac{1}{5}, \frac{2}{5})^\top$ when $m = 5$, and $\omega = (\frac{1}{10}, \frac{1}{10}, \frac{1}{10}, \frac{1}{10}, \frac{1}{20}, \frac{1}{20}, \frac{1}{20}, \frac{1}{20}, \frac{1}{5}, \frac{1}{5})^\top$ when $m = 10$.

The number of customers is chosen to range from $n = 50$ to $n = 500$ in incremental steps of 50. For each class of instances and each size of the problem we have generated 25 instances. All the runs were performed on a PC with a 1.2 GHz Pentium IV processor and

256 MB RAM. All LP-relaxations were solved using CPLEX 7.5 [4] and the heuristic was implemented in Microsoft Visual C.

5.1 Results for the base case

For the base case of $m = 5$ and $T = 6$, the results of the greedy heuristic and the local search procedure are presented in Tables 1–4. Tables 1–3 show the average time spent by the greedy heuristic, as well as an average error bound on the heuristic solution value (where the values presented correspond to averages over the 25 generated instances in each problem class). The error bound for a particular instance is given by

$$\text{error} = \frac{z_n^{\text{G}} - z_n^{\text{LP}}}{z_n^{\text{LP}}} \cdot 100\%,$$

i.e., we determine the relative error of the greedy objective value with respect to the optimal solution value of the LP-relaxation, which means that the actual errors could be much smaller than the ones shown. We can see that the errors are relatively small for large instances, however for small instances they can be quite large, in particular for $n = 50$. The fact that the upper bound on the error is smaller for the class of non-identical facilities, seasonal demand pattern and static customers than for any other class of problems could be explained by he fact that, for this case, the problem is a static SSP (see Proposition 3.4). The time consumed by the greedy heuristic is mainly used to solve the LP-relaxation of (P). We could always find a (complete) solution for $n \geq 150$. For the smaller instances with $n = 50$ and $n = 100$, Table 4 shows the number of infeasibilities. Not included in this table are three instances where the LP-relaxation itself was infeasible: one each for $n = 50$ in the static, the mixed and the dynamic case with non-identical facilities and seasonal demands. These results suggest that the greedy heuristic is asymptotically feasible, even for the classes of problems for which we were not able to prove this result. Finally, there seems to be no difference in difficulty between problem instances with identical and non-identical facilities.

Although the tables clearly illustrate that the relative error of the greedy solution decreases (to zero) as the number of customers increases, the error may be substantial for small values of n. We have performed two local exchange procedures to improve the greedy solution. When the greedy heuristic was not able to find a full assignment, we performed local exchanges between pairs of assignments to create enough space for customers (or (customer,period)-pairs) that were not assigned. This procedure was able to find a feasible solution in three out of the eight instances where the greedy heuristic failed to find one. To improve the value of the feasible solution, we performed a sequence of improving local exchanges between pairs of assignments. The results are shown in the same tables as the results of the greedy heuristics. The times shown for the local search exclude the time required to find the greedy solution.

5.2 Results with additional facilities or periods

The results of the greedy heuristic as well as the local search procedure for $m = 10$ and $T = 6$ are shown in Tables 5–8. The feasibility procedure was able to find a feasible solution except for three instances with $n = 50$ for the case of non-identical facilities and general demands, and three instances with $n = 50$ and one with $n = 100$ for the case of non-identical facilities and seasonal demands. Not included in Table 8 are three instances where the LP-relaxation itself was infeasible: one each for $n = 50$ in the static, the mixed and the dynamic case with non-identical facilities and seasonal demands. Tables 9–11 show all results for $m = 5$ and $T = 12$. In this case, the feasibility procedure was able to find a feasible solution in two out of the four instances where the greedy heuristic failed to find one.

From these experiments, we conclude that the errors of the greedy solutions are always decreasing in the number of customers, and appear to converge to zero for all cases. However, the relative errors of the greedy solution seem to increase with the number of facilities or the number of time periods when the number of customers is fixed. This is consistent with the result of Theorem 4.1, in which it is shown that the number of assignments in the greedy solution that is different from the assignments in the LP-solution is independent of the number of customers, but increases with the number of facilities and time periods. Recall also that the errors are measured with respect to the LP lower bound. This may seriously overestimate the actual errors when the integrality gap is substantial, which can be expected in particular when the ratio between the number of customers and the number of facilities is small.

	static				mixed				dynamic			
	greedy		LS		greedy		LS		greedy		LS	
n	time (sec.)	error (%)	time (sec.)	error (%)	time (sec.)	error (%)	time (sec.)	error (%)	time (sec.)	error (%)	time (sec.)	error (%)
50	0.05	10.22	0.00	3.85	0.05	16.87	0.00	3.97	0.05	15.87	0.02	2.49
100	0.11	5.42	0.01	1.21	0.11	8.03	0.05	1.53	0.10	9.59	0.12	1.29
150	0.16	4.35	0.05	0.93	0.16	6.88	0.11	1.09	0.16	6.41	0.31	0.84
200	0.22	3.42	0.09	0.64	0.23	6.12	0.21	0.89	0.23	4.77	0.55	0.72
250	0.32	1.57	0.10	0.33	0.34	4.72	0.30	0.79	0.34	3.25	0.89	0.38
300	0.40	2.56	0.17	0.45	0.39	3.43	0.46	0.62	0.39	3.17	1.32	0.41
350	0.48	1.57	0.23	0.22	0.49	3.02	0.66	0.40	0.48	2.33	1.62	0.24
400	0.60	1.59	0.30	0.22	0.62	2.74	0.83	0.37	0.57	2.27	2.37	0.26
450	0.72	1.62	0.39	0.22	0.76	2.34	1.09	0.35	0.81	2.12	2.83	0.25
500	0.86	0.94	0.45	0.15	0.84	1.96	1.30	0.32	0.87	1.79	3.39	0.22

Table 1: Greedy heuristic and local search (LS) for identical facilities and general demand pattern ($m = 5$, $T = 6$)

The computational times required to find a solution are insensitive to the number of facilities. The time required by the local search procedure does increase with the number of periods for the cases where some or all of the customers allow for dynamic assignments. This is caused by the fact that the total potential number of exchanges that needs to be considered increases quadratically in the number of time periods in these cases, while the number of exchanges is independent of T in the static case.

	static				mixed				dynamic			
	greedy		LS		greedy		LS		greedy		LS	
n	time (sec.)	error (%)	time (sec.)	error (%)	time (sec.)	error (%)	time (sec.)	error (%)	time (sec.)	error (%)	time (sec.)	error (%)
50	0.05	11.42	0.00	3.62	0.05	14.25	0.01	4.20	0.05	14.75	0.05	2.65
100	0.11	4.02	0.02	1.17	0.13	8.24	0.04	1.49	0.10	8.46	0.13	1.40
150	0.18	3.19	0.02	0.66	0.21	6.30	0.11	1.13	0.19	5.82	0.31	0.77
200	0.27	3.57	0.08	0.57	0.28	5.74	0.22	0.88	0.28	4.81	0.54	0.60
250	0.35	1.33	0.09	0.31	0.40	3.55	0.30	0.56	0.37	3.37	0.82	0.38
300	0.47	1.95	0.14	0.38	0.49	3.23	0.46	0.51	0.48	3.22	1.16	0.41
350	0.58	1.14	0.21	0.17	0.63	2.95	0.68	0.40	0.61	2.96	1.83	0.25
400	0.72	2.16	0.32	0.26	0.72	2.77	0.88	0.49	0.71	2.53	2.52	0.37
450	0.87	0.90	0.33	0.16	0.89	2.67	1.06	0.35	0.87	2.34	2.77	0.21
500	0.99	1.03	0.43	0.19	1.00	1.95	1.35	0.31	0.89	1.84	3.28	0.24

Table 2: Greedy heuristic and local search (LS) for non-identical facilities and general demand pattern ($m = 5$, $T = 6$)

	static				mixed				dynamic			
	greedy		LS		greedy		LS		greedy		LS	
n	time (sec.)	error (%)	time (sec.)	error (%)	time (sec.)	error (%)	time (sec.)	error (%)	time (sec.)	error (%)	time (sec.)	error (%)
50	0.05	5.60	0.00	1.63	0.04	15.07	0.01	4.23	0.05	14.09	0.01	2.98
100	0.11	3.03	0.00	0.84	0.11	6.78	0.05	1.56	0.10	8.70	0.12	1.27
150	0.16	1.83	0.03	0.43	0.16	5.12	0.10	0.82	0.17	5.24	0.26	0.61
200	0.22	1.29	0.08	0.29	0.27	5.04	0.20	0.75	0.25	4.05	0.49	0.51
250	0.30	1.52	0.10	0.25	0.33	4.16	0.29	0.53	0.30	3.57	0.77	0.44
300	0.38	0.90	0.15	0.13	0.40	2.50	0.39	0.32	0.42	2.55	1.12	0.24
350	0.48	0.57	0.18	0.11	0.52	2.66	0.58	0.35	0.56	2.07	1.51	0.25
400	0.59	1.02	0.27	0.15	0.61	2.81	0.78	0.34	0.64	2.58	2.12	0.30
450	0.70	1.23	0.37	0.13	0.66	3.00	1.10	0.35	0.62	2.57	2.72	0.27
500	0.82	0.63	0.43	0.10	0.85	1.56	1.19	0.17	0.95	1.56	3.07	0.17

Table 3: Greedy heuristic and local search (LS) for non-identical facilities and seasonal demand pattern ($m = 5$, $T = 6$)

	identical facilities general demands			non-identical facilities general demands			non-identical facilities seasonal demands		
n	static	mixed	dynamic	static	mixed	dynamic	static	mixed	dynamic
50	1	0	0	0	0	0	2	2	1
100	0	0	0	0	0	0	1	1	0

Table 4: Number of instances in which the greedy heuristic failed to find a full assignment ($m = 5$, $T = 6$)

	static				mixed				dynamic			
	greedy		LS		greedy		LS		greedy		LS	
n	time (sec.)	error (%)	time (sec.)	error (%)	time (sec.)	error (%)	time (sec.)	error (%)	time (sec.)	error (%)	time (sec.)	error (%)
50	0.10	18.79	0.00	7.53	0.10	37.20	0.01	13.30	0.09	37.78	0.03	8.21
100	0.22	18.90	0.05	5.05	0.22	24.88	0.06	5.40	0.17	25.48	0.17	3.75
150	0.38	12.96	0.06	2.82	0.38	18.91	0.11	3.81	0.32	16.72	0.35	2.62
200	0.58	8.13	0.09	1.57	0.56	14.59	0.26	2.43	0.46	12.62	0.72	1.57
250	0.81	8.49	0.16	1.61	0.70	14.08	0.43	2.25	0.59	12.86	1.07	1.70
300	1.07	7.34	0.22	1.43	0.98	10.74	0.57	1.98	0.84	9.90	1.83	1.23
350	1.39	5.58	0.31	0.90	1.24	8.81	0.83	1.36	1.07	8.37	2.37	0.97
400	1.70	4.64	0.40	0.69	1.47	7.75	1.11	1.29	1.22	6.74	3.04	0.82
450	2.06	4.48	0.49	0.70	1.82	7.37	1.40	1.12	1.55	6.10	3.66	0.67
500	2.44	3.33	0.68	0.59	2.11	5.71	1.85	0.89	1.83	5.48	4.65	0.62

Table 5: Greedy heuristic and local search (LS) for identical facilities and general demand pattern ($m = 10$, $T = 6$)

	static				mixed				dynamic			
	greedy		LS		greedy		LS		greedy		LS	
n	time (sec.)	error (%)	time (sec.)	error (%)	time (sec.)	error (%)	time (sec.)	error (%)	time (sec.)	error (%)	time (sec.)	error (%)
50	0.11	16.57	0.01	8.43	0.11	36.01	0.03	14.51	0.08	41.38	0.04	11.85
100	0.26	15.90	0.02	4.10	0.27	25.26	0.06	6.18	0.24	25.57	0.20	3.61
150	0.43	11.61	0.05	2.38	0.46	21.18	0.15	4.10	0.38	17.26	0.39	2.46
200	0.67	6.82	0.09	1.28	0.68	14.06	0.26	2.17	0.71	11.40	0.78	1.47
250	0.91	9.62	0.15	1.72	0.85	13.64	0.43	2.32	0.88	11.61	1.25	1.47
300	1.20	6.73	0.23	1.22	1.22	10.70	0.68	1.82	1.14	9.58	1.89	1.05
350	1.58	4.61	0.30	0.87	1.56	8.43	0.85	1.36	1.39	8.28	2.38	0.93
400	1.97	3.81	0.40	0.60	1.87	6.77	1.13	1.22	1.65	6.52	3.24	0.74
450	2.44	5.47	0.59	0.79	2.38	7.30	1.54	1.05	1.93	5.65	4.09	0.59
500	2.88	3.43	0.65	0.52	2.77	4.88	1.86	0.77	2.58	4.67	4.74	0.52

Table 6: Greedy heuristic and local search (LS) for non-identical facilities and general demand pattern ($m = 10$, $T = 6$)

	static				mixed				dynamic			
	greedy		LS		greedy		LS		greedy		LS	
n	time (sec.)	error (%)	time (sec.)	error (%)	time (sec.)	error (%)	time (sec.)	error (%)	time (sec.)	error (%)	time (sec.)	error (%)
50	0.10	11.68	0.00	6.12	0.10	33.59	0.01	13.29	0.08	35.47	0.03	9.71
100	0.26	9.67	0.01	2.25	0.27	22.12	0.05	5.36	0.22	22.74	0.16	3.28
150	0.49	8.36	0.04	1.57	0.47	16.84	0.15	3.75	0.40	17.27	0.40	2.60
200	0.86	5.63	0.11	1.05	0.77	13.98	0.29	2.48	0.64	13.94	0.81	1.73
250	0.97	4.26	0.15	0.70	0.89	11.26	0.51	2.07	0.76	10.38	1.17	1.35
300	1.25	3.21	0.20	0.55	1.23	10.77	0.65	1.49	1.21	9.36	1.75	1.06
350	1.50	3.94	0.31	0.63	1.46	9.39	0.93	1.28	1.27	7.48	2.20	0.88
400	1.88	2.11	0.39	0.41	1.96	7.70	1.27	1.06	1.66	6.92	2.93	0.78
450	2.50	1.87	0.54	0.33	2.19	6.43	1.56	0.92	1.95	6.05	4.10	0.70
500	3.02	1.91	0.65	0.32	2.47	7.02	2.04	0.92	2.41	5.98	5.86	0.69

Table 7: Greedy heuristic and local search (LS) for non-identical facilities and seasonal demand pattern ($m = 10$, $T = 6$)

	identical facilities general demands			non-identical facilities general demands			non-identical facilities seasonal demands		
n	static	mixed	dynamic	static	mixed	dynamic	static	mixed	dynamic
50	3	0	0	12	0	0	8	0	0
100	0	0	0	0	0	0	4	0	0

Table 8: Number of instances in which the greedy heuristic failed to find a full assignment ($m = 10$, $T = 6$)

	static				mixed				dynamic			
	greedy		LS		greedy		LS		greedy		LS	
n	time (sec.)	error (%)	time (sec.)	error (%)	time (sec.)	error (%)	time (sec.)	error (%)	time (sec.)	error (%)	time (sec.)	error (%)
50	0.10	16.35	0.00	4.25	0.10	25.79	0.04	4.80	0.11	27.52	0.12	3.00
100	0.25	8.99	0.02	1.76	0.27	18.35	0.17	2.84	0.29	14.00	0.47	1.62
150	0.43	4.94	0.06	0.96	0.46	8.95	0.34	1.51	0.52	7.71	1.08	0.85
200	0.63	3.91	0.12	0.70	0.72	7.35	0.64	1.09	0.91	6.94	2.16	0.73
250	0.85	3.81	0.20	0.67	0.86	6.08	1.03	0.91	0.92	6.06	3.57	0.66
300	1.12	2.41	0.28	0.39	1.28	4.87	1.42	0.71	1.64	4.43	4.87	0.45
350	1.45	2.16	0.42	0.26	1.58	3.66	2.01	0.52	2.31	3.58	8.58	0.27
400	1.82	2.23	0.50	0.29	1.92	3.79	2.69	0.56	2.52	3.07	15.71	0.25
450	2.23	1.25	0.59	0.18	2.59	3.89	3.21	0.35	3.90	2.81	26.20	0.21
500	2.72	1.38	0.81	0.17	3.06	3.12	4.05	0.46	4.95	2.94	34.31	0.21

Table 9: Greedy heuristic and local search (LS) for identical facilities and general demand pattern ($m = 5$, $T = 12$)

	static				mixed				dynamic			
	greedy		LS		greedy		LS		greedy		LS	
n	time (sec.)	error (%)	time (sec.)	error (%)	time (sec.)	error (%)	time (sec.)	error (%)	time (sec.)	error (%)	time (sec.)	error (%)
50	0.12	15.67	0.00	3.73	0.12	25.73	0.03	5.25	0.13	25.35	0.11	2.99
100	0.26	7.62	0.02	1.49	0.30	15.18	0.15	2.59	0.29	13.97	0.47	1.64
150	0.45	6.01	0.07	1.13	0.52	10.08	0.38	1.48	0.54	8.53	1.09	0.79
200	0.67	4.90	0.13	0.85	0.86	8.06	0.63	1.18	1.02	7.12	1.91	0.66
250	0.99	3.27	0.17	0.63	1.15	6.62	1.00	0.76	1.03	5.68	3.25	0.55
300	1.29	1.98	0.28	0.37	1.53	5.51	1.45	0.84	1.73	4.83	4.55	0.42
350	1.69	1.55	0.38	0.25	1.75	3.95	1.81	0.54	2.38	3.62	7.53	0.25
400	2.12	1.68	0.43	0.22	2.03	3.64	2.32	0.48	2.90	3.14	14.10	0.26
450	2.58	1.43	0.62	0.17	2.69	4.15	3.17	0.41	4.19	3.29	22.98	0.22
500	3.11	1.62	0.83	0.18	3.26	3.28	3.79	0.39	5.80	2.85	33.45	0.19

Table 10: Greedy heuristic and local search (LS) for non-identical facilities and general demand pattern ($m = 5, T = 12$)

	static				mixed				dynamic			
	greedy		LS		greedy		LS		greedy		LS	
n	time (sec.)	error (%)	time (sec.)	error (%)	time (sec.)	error (%)	time (sec.)	error (%)	time (sec.)	error (%)	time (sec.)	error (%)
50	0.07	9.57	0.01	1.74	0.12	20.60	0.03	4.37	0.13	20.83	0.10	2.67
100	0.23	4.68	0.02	0.65	0.28	14.41	0.15	2.59	0.32	14.36	0.46	1.59
150	0.39	2.99	0.05	0.36	0.49	10.48	0.33	1.20	0.57	9.73	1.04	0.94
200	0.58	2.31	0.12	0.27	0.71	7.57	0.57	0.90	0.79	6.91	1.96	0.65
250	0.81	1.74	0.17	0.21	0.99	6.71	0.96	0.70	1.09	5.77	2.77	0.51
300	1.09	1.08	0.24	0.14	1.35	5.84	1.36	0.52	1.91	4.59	4.29	0.37
350	1.39	0.99	0.30	0.15	1.95	5.15	1.81	0.48	2.93	4.32	8.51	0.34
400	1.76	0.85	0.39	0.10	2.10	4.59	2.63	0.42	2.75	3.70	15.90	0.30
450	2.15	0.84	0.51	0.10	2.77	4.15	3.11	0.34	3.89	3.36	24.38	0.25
500	2.61	0.70	0.70	0.08	3.06	3.26	3.89	0.22	4.59	2.70	34.35	0.16

Table 11: Greedy heuristic and local search (LS) for non-identical facilities and seasonal demand pattern ($m = 5$, $T = 12$)

	identical facilities general demands			non-identical facilities general demands			non-identical facilities seasonal demands		
n	static	mixed	dynamic	static	mixed	dynamic	static	mixed	dynamic
50	0	0	0	0	0	0	3	0	1

Table 12: Number of instances in which the greedy heuristic failed to find a full assignment ($m = 10$, $T = 12$)

6 Summary and future research

In this paper we have considered the problem of evaluating a logistics network design in a dynamic environment. We have proposed a new class of pseudo-cost functions for the greedy heuristic that was developed by Martello and Toth [12] for the GAP, and have shown that a particular element from that class yields a greedy heuristic that is asymptotically optimal in a probabilistic sense for two large subclasses of the problem, namely problems where the assignment of customers to facilities is allowed to vary over time, as well as problems where each customer needs to be assigned to the same facility over the planning horizon, and in addition the demand for each customer exhibits the same seasonality pattern. This behavior is illustrated with some numerical results of the greedy heuristic. The results obtained suggest that the greedy heuristic may be asymptotically optimal even for the cases that we were unable to analyze theoretically. In addition, it is shown that significant improvements can be made by using the result of the greedy heuristic as the starting point of a local interchange procedure, yielding very nearly optimal solutions for problems with many customers. We are currently investigating an extension of the model with limited throughput and physical inventory capacities for each warehouse, as well as perishability constraints (see Ahuja et al. [1]). Other interesting extensions of the models would deal with more echelons (see for instance Romeijn and Romero Morales [19] for computational results), and multiple products.

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Appendix

Theorem 4.1 *There exists some constant R, independent of n, so that $|N| \leq R$ for all instances of (LP) that are feasible and non-degenerate.*

Proof: We will first prove the result for the purely static case, i.e., $\mathcal{D} = \emptyset$. In that case, the set N reads

$$N = \{j = 1, \ldots, n : x^{\mathrm{G}}_{\cdot j1} \neq x^{\mathrm{LP}}_{\cdot j1}\}.$$

It is obvious that it would be possible to fix all feasible assignments from x^{LP} without violating any capacity constraint. Proposition 2.2 ensures that the most desirable facility for each customer that is feasibly assigned in x^{LP} is equal to the facility to which it is assigned in x^{LP}. Moreover, the same proposition shows that the initial desirabilities are such that the greedy heuristic starts by assigning customers that are feasibly assigned in x^{LP}. Now suppose that the greedy heuristic would reproduce all the assignments that are feasible in x^{LP}. Then, because the remaining assignments in x^{LP} are infeasible with respect to the integrality constraints, x^{G} and x^{LP} would differ only in those last ones. By Lemma 2.1 we know that then $|N| \leq mT$, and the result follows. So in the remainder of the proof we will assume that x^{G} and x^{LP} differ in at least one assignment that is feasible in the latter.

However, while the greedy heuristic is assigning customers that are feasibly assigned in x^{LP} it may at some point start updating the desirabilities of the assignments still to be made due to the decreasing remaining capacities. This may cause the greedy heuristic to deviate from one of the feasible assignments in x^{LP}. Such an assignment could use some capacity that x^{LP} uses for other (feasible) assignments. In particular, this assignment uses at most $t\overline{D}$ units of capacity through period t, for each $t = 1, \ldots, T$. Since the facility that is involved in this assignment may now not be able to accommodate all customers that were feasibly assigned to it in x^{LP}, other deviations from the feasible assignments in x^{LP} will occur. Since any other assignment requires at least $t\overline{D}$ units of capacity through period t for each $t = 1, \ldots, T$, the number of additional deviations is at most equal to $\max_{t=1,\ldots,T} \lceil (t\overline{D})/(t\underline{D}) \rceil = \lceil \overline{D}/\underline{D} \rceil$. In the remainder of this proof we will show that the total number of deviations is bounded by a constant independent of n. In order to make this precise, we will first bound the number of times that the desirabilities ρ must be recalculated, and then bound the number of deviations from x^{LP} between these recalculations.

The calculation of the values of ρ depends only on the set of feasible facilities for each $(j, t) \in L$. A facility can always feasibly supply any customer if its cumulative capacity through period t is at least $t\overline{D}$ for all $t \in \{1, \ldots, T\}$. The values of ρ therefore only may need to be recalculated when the cumulative capacity for a facility through period t is below $t\overline{D}$ for some $t \in \{1, \ldots, T\}$. Since the cumulative demand for each customer through period t is at least $t\underline{D}$, this happens at most $\lceil t\overline{D}/t\underline{D} \rceil = \lceil \overline{D}/\underline{D} \rceil$ for each facility in each time period, and thus the number of times that the desirabilities ρ must be recalculated is no more than $mT \lceil \overline{D}/\underline{D} \rceil$.

Now let $l^{(k)}$ be the iteration that induces the k-th recalculation of the values of the desirabilities ρ, and assume that this recalculation has taken place. Let M^k be the set of customers that have been assigned in the first $l^{(k)}$ iterations and do not coincide with x^{LP}. Let U^k be the set of customers that have not been assigned in the first $l^{(k)}$ iterations and for which we would get a different assignment than in x^{LP} by assigning them to their current most desirable facility (thus, if $j \in U^k$ then $x^{\text{LP}}_{i_j,j,1} \neq 1$). In other words, U^k contains the customers that have not been assigned in the first $l^{(k)}$ iterations, and that would belong to N if they were assigned to their most desirable facility.

First note that Proposition 2.2 ensures that initially the most desirable facility in our greedy heuristic for each $j \notin B_{\mathcal{S}}$ coincides with the corresponding assignment in x^{LP}. Moreover, in the original ordering of the desirabilities, we first encounter all customers not in $B_{\mathcal{S}}$, followed by all customers in $B_{\mathcal{S}}$. Since x^{G} and x^{LP} do not coincide for at least one customer that is feasibly assigned in x^{LP}, $|M^1| = 0$ and the set of customers not assigned in the first $l^{(1)}$ iterations for which the most desirable facility does not coincide with the corresponding assignment in x^{LP} is a subset of the set of infeasible assignments in x^{LP}, thus

$$|U^1| \leq |B_{\mathcal{S}}| \leq mT,$$

by using Lemma 2.1. It is easy to see that, for $k \geq 1$, the number of customers that have been assigned in the first $l^{(k+1)}$ iterations and do not coincide with x^{LP} is at most equal to the number of customers that have been assigned in the first $l^{(k)}$ iterations and do not coincide with x^{LP}, plus the number of customers that would be assigned to a facility not coinciding with x^{LP} if they were assigned in one of the iterations $l^{(k)} + 1, \ldots, l^{(k+1)}$. In other words,

$$|M^{k+1}| \leq |M^k| + |U^k|. \tag{10}$$

Moreover, the assignments made in the last $l^{(k+1)} - l^{(k)}$ iterations that were different from the corresponding assignment in x^{LP} could each cause additional deviations from x^{LP}. In particular, each of these assignments could cause at most $\lceil \overline{D}/\underline{D} \rceil$ assignments still to be made to deviate from x^{LP}. Thus,

$$
\begin{aligned}
|U^{k+1}| &\leq |U^k| + (|M^{k+1}| - |M^k|) \left\lceil \frac{\overline{D}}{\underline{D}} \right\rceil \\
&\leq |U^k| + |M^{k+1}| \left\lceil \frac{\overline{D}}{\underline{D}} \right\rceil \\
&\leq |U^k| + (|M^k| + |U^k|) \left\lceil \frac{\overline{D}}{\underline{D}} \right\rceil \qquad \text{using equation (10)} \\
&\leq (|M^k| + |U^k|) \left(1 + \left\lceil \frac{\overline{D}}{\underline{D}} \right\rceil \right).
\end{aligned}
$$

Using induction, it can now be shown that

$$
\begin{aligned}
|M^k| &\leq mT \left(2 + \left\lceil \frac{\overline{D}}{\underline{D}} \right\rceil \right)^{k-2} \\
|U^k| &\leq mT \left(2 + \left\lceil \frac{\overline{D}}{\underline{D}} \right\rceil \right)^{k-1}
\end{aligned}
$$

for each k.

If the number of times the desirabilities are recalculated is equal to k^*, then $N \subseteq M^{k^*} \cup U^{k^*}$, and thus $|N| \leq |M^{k^*+1}|$. Now recall that $k^* \leq mT \left\lceil \overline{D}/\underline{D} \right\rceil$. Thus, $|N|$ is bounded from above by a constant not depending on n, yielding the desired result.

The proof for the general case is similar to the proof of the static case. Basically, we again need to bound the number of times the desirabilities ρ must be recalculated, and the number of affected assignments by an assignment that is different from an assignment in x^{LP}.

The feasibility of a facility is an issue only when its cumulative capacity through period t is below $t\overline{D}$ (for static customers) or $\overline{D}$ (for dynamic customers). Then, it is easy to see that ρ only needs to be recalculated when after an assignment the cumulative capacity through period t of the used facility is below $t\overline{D}$ for one or more values of $t = 1, \ldots, T$. Moreover, a static assignment uses at least $t\underline{D}$ units of cumulative capacity in each period t, while a dynamic assignment uses at least $\underline{D}$ units of capacity in at least one period. Thus, ρ must be recalculated at most $mT \left\lceil T\overline{D}/\underline{D} \right\rceil$ times.

An assignment in x^{G} that is different from the corresponding assignment in x^{LP} uses at most $t\overline{D}$ units of cumulative capacity through period t for all $t = 1, \ldots, T$ that x^{LP} uses for other assignments. Since the minimal demand is bounded from below by $\underline{D}$, an upper bound on the number of possible affected assignments is $\left\lceil T\overline{D}/\underline{D} \right\rceil$.

The desired result now easily follows in a similar way as for the static case. □

Theorem 4.5 *Under Assumption 3.6, the greedy heuristic given in Section 4.1 combined with the local exchange procedure for feasibility is asymptotically* feasible *with probability one.*

Proof: Using the same arguments as in Theorem 4.3, we know that (LP) is non-degenerate with probability one and feasible with probability one when $n \to \infty$. It suffices to show that the local exchange procedure for feasibility applied to the partial solution obtained by the greedy heuristic, x^{G}, is asymptotically feasible. In particular, we will derive a set of sufficient conditions under which the local exchange procedure applied to the greedy solution will find a feasible solution. We will then show that this set of conditions is satisfied with probability one when $n \to \infty$.

After applying the greedy heuristic, let $\mathcal{U}$ again denote the set of unassigned (customer, period)-pairs. Recall that the local exchange procedure for feasibility considers the pairs in $\mathcal{U}$ for assignment in increasing order of the period to which they belong. Now let $\mathcal{U}_t$ be the set of unassigned pairs after the local exchange procedure for feasibility has considered all pairs from periods 1 till t ($t = 1, \ldots, T$), and define $\mathcal{U}_0 \equiv \mathcal{U}$. Consider some period t ($t = 1, \ldots, T$). To assign any pair in that period, it is easy to see that we have to unassign at most $\lceil \overline{D}/\underline{D} \rceil$ pairs in future periods. So each element of $\mathcal{U}_t$ that gets assigned in period t yields at most $\lceil \overline{D}/\underline{D} \rceil$ pairs in future periods that need to be assigned, and each element of $\mathcal{U}_t$ that does not correspond to period t simply remains to be assigned in a future period. This implies that

$$|\mathcal{U}_{t+1}| \leq \lceil \overline{D}/\underline{D} \rceil \, |\mathcal{U}_t|$$

for $t = 1, \ldots, T$. Using the fact that $|\mathcal{U}| \leq |N|$, it is easy to see that $|\mathcal{U}_t|$ is bounded from above by a constant independent of n (see Theorem 4.1).

Now consider the first period. The set of pairs that remain to be assigned to a facility in this period is equal to $\mathcal{U}_0$. Recall that the local exchange for feasibility is able to assign a pair $(\hat{\jmath}, 1) \in \mathcal{U}_0$ to facility $\hat{\imath}$ if

$$b_{\hat{\imath}1} - \sum_{j=1}^{n} d_{j1} x^{\mathrm{G}}_{\hat{\imath}j1} > d_{\hat{\jmath}1}.$$

Such a facility exists for all customers that remain to be assigned in period 1 if

$$\sum_{i=1}^{m} b_{i1} - \sum_{i=1}^{m} \sum_{j=1}^{n} d_{j1} x^{\mathrm{G}}_{ij1} > m\overline{D} \, \max\{2, |\mathcal{U}_0|\} \equiv K_1.$$

Similarly, it can be shown for $\tau = 2, \ldots, T$ that

$$\sum_{t=1}^{\tau} \sum_{i=1}^{m} b_{it} - \sum_{t=1}^{\tau} \sum_{i=1}^{m} \sum_{j=1}^{n} d_{jt} x^{\mathrm{G}}_{ijt} > m\overline{D} \max\{2, \sum_{t=1}^{\tau} |\mathcal{U}_{t-1}|\} \equiv K_\tau$$

implies that all pairs in $\mathcal{U}_\tau$ can be assigned.

It is now easy to see that each K_t can be bounded from above by a constant independent of n. This, together with Assumption 3.6 implies that the necessary capacities are indeed present with probability one if $n \to \infty$. □