# An asymptotically optimal greedy heuristic for the multi-period single-sourcing problem: the cyclic case*

H. Edwin Romeijn† Dolores Romero Morales‡

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**Abstract**

The dynamics of the environment in which supply chains evolve requires that companies frequently redesign their logistics distribution networks. In this paper we address a multi-period single-sourcing problem that can be used as a strategic tool for evaluating the costs of logistics network designs in a dynamic environment. The distribution networks that we consider consist of a set of production and storage facilities, and a set of customers who do not hold inventories. The facilities face production capacities, and each customer's demand needs to be delivered by a single facility in each period. We deal with the assignment of customers to facilities, as well as the location, timing, and size of inventories. In addition, to mitigate start and end-of-study effects, we view the planning period as a typical future one, which will repeat itself. This leads to a cyclic model, in which starting and ending inventories are equal. Based on an assignment formulation of the problem, we propose a greedy heuristic, and prove that this greedy heuristic is asymptotically feasible and optimal in a probabilistic sense. We illustrate the behavior of the greedy heuristic, as well as some improvements where the greedy heuristic is used as the starting point of a local interchange procedure, on a set of randomly generated test problems.

## 1 Introduction

The tendency to move towards global supply chains and the shortening of the product life cycle cause companies to consider redesigning their logistics distribution network. Most of the quantitative models proposed in the literature for the strategic problem of evaluating (usually with respect to costs) the design (e.g. the location and capacities of facilities) of a logistics distribution network assume a static environment. Hence the adequacy of those

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†Department of Industrial and Systems Engineering, University of Florida, 303 Weil Hall, P.O. Box 116595, Gainesville, Florida 32611-6595; email: romeijn@ise.ufl.edu.

‡Faculty of Economics and Business Administration, Maastricht University, P.O. Box 616, 6200 MD Maastricht, The Netherlands; e-mail: d.romero@ke.unimaas.nl. The work of this author was supported in part by the Netherlands Organization for Scientific Research (NWO).

models is limited to situations where, in particular, the demand pattern is stationary over time. In addition, inventory decisions cannot be supported using stationary models.

In this paper we will study a multi-period single-sourcing problem (MPSSP) that can be used for evaluating logistics distribution network designs with respect to costs in a dynamic environment. The logistics distribution network consists of a set of facilities where production and storage take place, as well as a set of customers. For a given planning horizon, the customers' demand patterns for a single product are assumed known. Since this model is intended to be used for strategic purposes, we assume that the planning period is a typical future one, and will repeat itself over time. In other words, the model is cyclic in nature. We assume that there is no transportation between the facilities, but only between the facilities and the customers. We assume that these transports are carried out by a third-party logistics provider, or that each customer receives an individual shipment. In addition, we do not allow for inventories at the customers. This situation is typical in, for instance, the food and beverage industry, where the customers often are supermarkets and restaurants, which usually have very limited storage capacity.

The decisions that need to be made are (i) the assignment of customers to facilities, and (ii) the location and size of inventories at the facilities. We assume that each facility has known, finite, and possibly time-varying, production capacity. Moreover, we assume that each facility has essentially unlimited physical storage and throughput capacity. In other words, we assume that its physical storage capacity is sufficient to be able to store the cumulative excess production of the facility, even if it produces to full capacity in each period. In addition, the throughput capacity is large enough for the facilities to be able to supply any combination of customers assigned to it. The demand of each customer needs to be delivered by a single facility in each period. Finally, we assume linear production and inventory holding costs. However, the assignment nature of the model allows for transportation cost functions that are arbitrary functions of demand and distance. As we will show, this problem can be formulated as a mixed-integer linear programming problem.

Since even the problem of determining whether there exists a feasible solution to the MPSSP is NP-complete (see Martello and Toth [9] and Romero Morales, Van Nunen and Romeijn [14]), it is unlikely that efficient methods exist that can solve large problem instances to optimality. Therefore, it is appropriate to study heuristic approaches to this problem. To this end, we will reformulate the MPSSP as a Generalized Assignment Problem (GAP) with a nonlinear objective function. We will then propose a new family of pseudo-cost functions for the class of greedy heuristics for the GAP proposed by Martello and Toth [8], in the same spirit as the family of pseudo-cost functions for the GAP in Romeijn and Romero Morales [11]. In the latter paper, the optimal solution vector of the dual programming problem corresponding to the linear programming relaxation of the GAP is used to define a pseudo-cost function that yields a greedy heuristic that is asymptotically feasible and optimal in a probabilistic sense. However, due to the nonlinearity of the objective function in the GAP formulation of the MPSSP, the pseudo-cost function that is suitable for the (linear) GAP is not defined for the MPSSP. In this paper, we will derive an adequate pseudo-cost function for the MPSSP for which the corresponding greedy heuristic is asymptotically feasible and optimal in a probabilistic sense.

As mentioned above, related literature focuses mainly on static models. Examples are Geoffrion and Graves [7], Benders et al. [1], and Fleischmann [6]. Duran [4] studies a dynamic model for the planning of production, bottling, and distribution of beer, but focuses on the production process. Chan, Muriel and Simchi-Levi [2] study a dynamic, but uncapacitated, distribution problem in an operational setting.

The remainder of the paper is organized as follows. In Section 2 we will formulate the multi-period single-sourcing problem as a mixed-integer linear programming problem, derive some properties of its LP-relaxation, and show the relationship with the GAP through a reformulation of the problem as a pure assignment problem with a nonlinear objective function. In Section 3 we will probabilistically analyze the problem. In Section 4 we will introduce a class of greedy heuristics for the problem, and study the asymptotic behavior of a particular element from that class. Numerical experiments will be presented in Section 5, for the greedy heuristic as well as for two local exchange procedures for improving a given (partial) solution to the assignment problem. The paper ends in Section 6 with some concluding remarks.

# 2 The multi-period single-sourcing problem

## 2.1 A mixed-integer formulation

Let $n$ denote the number of customers, $m$ the number of facilities, and $T$ the length of the planning horizon. The demand of customer $j$ in period $t$ is given by $d_{jt}$, while the production capacity at facility $i$ in period $t$ is equal to $b_{it}$. The unit production costs at facility $i$ in period $t$ are $p_{it}$, and the costs of assigning customer $j$ to facility $i$ in period $t$ are $a_{ijt}$. Note that the assignment costs can be arbitrary functions of demand and distance. Unit inventory holding costs at facility $i$ in period $t$ are equal to $g_{it}$. We will make the common assumption that there are no speculative motives for production, i.e., $p_{it} + g_{it} \geq p_{i,t+1}$ for each $t = 1, \ldots, T-1$ and $i = 1, \ldots, m$. As mentioned in the introduction, we will assume that the planning horizon of length $T$ is a typical one, that will repeat itself over time. In particular, all problem data are assumed cyclic with cycle length $T$. For example, $d_{j,T+1} = d_{j1}$, $d_{j,T+2} = d_{j2}$, .... As a consequence, the inventory pattern at the facilities will (without loss of optimality) be cyclic as well. This can be modeled by letting the initial inventory level be equal to the final inventory level. Customer service considerations may necessitate that some or all customers are assigned to the same facility in each period. To incorporate this possibility into the model, we introduce the set $\mathcal{S} \subseteq \{1, \ldots, n\}$ of customers (called static customers) that needs to be assigned to the same facility in all periods. We let $\mathcal{D} = \{1, \ldots, n\} \backslash \mathcal{S}$ denote the remaining set of customers (called dynamic customers).

The problem can now be formulated as follows:

$$\text{minimize} \sum_{t=1}^{T} \sum_{i=1}^{m} p_{it} y_{it} + \sum_{t=1}^{T} \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ijt} x_{ijt} + \sum_{t=1}^{T} \sum_{i=1}^{m} g_{it} I_{it}$$

subject to (P$_0$)

$$y_{it} \leq b_{it} \qquad i = 1, \ldots, m;\ t = 1, \ldots, T$$

$$
\begin{aligned}
\sum_{j=1}^{n} d_{jt}x_{ijt} + I_{it} &= y_{it} + I_{i[t-1]} && i = 1,\ldots,m;\ t = 1,\ldots,T \\
\sum_{i=1}^{m} x_{ijt} &= 1 && j = 1,\ldots,n;\ t = 1,\ldots,T \\
y_{it} &\geq 0 && i = 1,\ldots,m;\ t = 1,\ldots,T \\
x_{ijt} &= x_{ij1} && i = 1,\ldots,m;\ j \in \mathcal{S};\ t = 2,\ldots,T \\
x_{ijt} &\in \{0,1\} && i = 1,\ldots,m;\ j = 1,\ldots,n;\ t = 1,\ldots,T \\
I_{it} &\geq 0 && i = 1,\ldots,m;\ t = 1,\ldots,T
\end{aligned}
$$

where $y_{it}$ denotes the quantity produced at facility $i$ in period $t$, $x_{ijt} = 1$ if customer $j$ is assigned to facility $i$ in period $t$ and 0 otherwise, and $I_{it}$ denotes ending inventory at facility $i$ at the end of period $t$. For convenience, we have introduced the notation $[t] = (t+1) \bmod T - 1$, i.e., $I_{i[t-1]} = I_{i,t-1}$ for $t = 2,\ldots,T$, and $I_{i[0]} = I_{iT}$. We simplify this problem by eliminating the production variables $y_{it}$. Using the absence of speculative motives in the production, we obtain the following reformulation of the problem, which we will call the *multi-period single-sourcing problem (MPSSP)*:

$$
\text{minimize } \sum_{t=1}^{T}\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ijt}x_{ijt} + \sum_{t=1}^{T}\sum_{i=1}^{m} h_{it}I_{it}
$$

subject to (P)

$$
\begin{aligned}
\sum_{j=1}^{n} d_{jt}x_{ijt} + I_{it} &\leq b_{it} + I_{i[t-1]} && i = 1,\ldots,m;\ t = 1,\ldots,T \\
\sum_{i=1}^{m} x_{ijt} &= 1 && j = 1,\ldots,n;\ t = 1,\ldots,T \\
x_{ijt} &= x_{ij1} && i = 1,\ldots,m;\ j \in \mathcal{S};\ t = 2,\ldots,T \\
x_{ijt} &\in \{0,1\} && i = 1,\ldots,m;\ j = 1,\ldots,n;\ t = 1,\ldots,T \\
I_{it} &\geq 0 && i = 1,\ldots,m;\ t = 1,\ldots,T
\end{aligned}
$$

where

$$c_{ijt} = a_{ijt} + p_{it}d_{jt}$$

and

$$h_{it} = p_{it} - p_{i[t+1]} + g_{it}.$$

We may observe that the new unit inventory holding costs are nonnegative due to the absence of speculative motives. This fact renders the nonnegativity constraints $y_{it} \geq 0$ redundant, and hence their equivalent in terms of the variables $x_{ijt}$ and $I_{it}$ has been omitted from (P). In the remainder of the paper we will use the term *assignment* to refer to both a static customer and a (customer,period)-pair when the customer is dynamic.

In the following section, we will derive some properties of the LP-relaxation of this problem and its dual.

## 2.2 Properties of the LP-relaxation of the MPSSP

The LP-relaxation of the MPSSP can be obtained by replacing the Boolean constraints on $x_{ijt}$ by nonnegativity constraints:

$$\text{minimize} \sum_{t=1}^{T}\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ijt}x_{ijt} + \sum_{t=1}^{T}\sum_{i=1}^{m} h_{it}I_{it}$$

subject to (LP)

$$\begin{aligned}
\sum_{j=1}^{n} d_{jt}x_{ijt} + I_{it} &\leq b_{it} + I_{i[t-1]} && i=1,\ldots,m;\ t=1,\ldots,T && (1)\\
\sum_{i=1}^{m} x_{ijt} &= 1 && j=1,\ldots,n;\ t=1,\ldots,T && (2)\\
x_{ijt} &= x_{ij1} && i=1,\ldots,m;\ j\in\mathcal{S};\ t=2,\ldots,T && (3)\\
x_{ijt} &\geq 0 && i=1,\ldots,m;\ j=1,\ldots,n;\ t=1,\ldots,T\\
I_{it} &\geq 0 && i=1,\ldots,m;\ t=1,\ldots,T.
\end{aligned}$$

The following lemma derives a bound on the number of split assignments in an optimal basic solution of (LP), i.e., the number of assignments that are infeasible with respect to the integrality constraints of (P). For that, we need to introduce some notation. Let $B_{\mathcal{S}}$ be the set of customers in $\mathcal{S}$ such that $j \in B_{\mathcal{S}}$ means that customer $j$ is split (i.e., customer $j$ is assigned to more than one facility, each satisfying part of its demand), and $B_{\mathcal{D}}$ be the set of (customer,period)-pairs such that $(j,t) \in B_{\mathcal{D}}$ means that customer $j \in \mathcal{D}$ is split in period $t$.

**Lemma 2.1** *If (LP) is feasible, a basic optimal solution of (LP) satisfies:*

$$|B_{\mathcal{S}}| + |B_{\mathcal{D}}| \leq mT.$$

**Proof:** We know that the total number of assignments to be made is equal to $|\mathcal{S}| + T|\mathcal{D}|$. It is clear that

$$|\mathcal{S} \setminus B_{\mathcal{S}}| + |\{1,\ldots,T\} \times \mathcal{D} \setminus B_{\mathcal{D}}| + |B_{\mathcal{S}}| + |B_{\mathcal{D}}| = |\mathcal{S}| + T|\mathcal{D}|. \quad (4)$$

The number of nonzero assignment variables is at least equal to

$$|\mathcal{S} \setminus B_{\mathcal{S}}| + |\{1,\ldots,T\} \times \mathcal{D} \setminus B_{\mathcal{D}}| + 2\cdot(|B_{\mathcal{S}}| + |B_{\mathcal{D}}|),$$

and we know that the total number of nonzero variables in the model is at most equal to the number of constraints in the model (after substituting the constraints (3) and eliminating the corresponding variables from the model), i.e., $mT + |\mathcal{S}| + T|\mathcal{D}|$, which yields the inequality

$$|\mathcal{S} \setminus B_{\mathcal{S}}| + |\{1,\ldots,T\} \times \mathcal{D} \setminus B_{\mathcal{D}}| + 2\cdot(|B_{\mathcal{S}}| + |B_{\mathcal{D}}|) \leq mT + |\mathcal{S}| + T|\mathcal{D}|. \quad (5)$$

The desired result then follows by combining (4) and (5). □

After eliminating the variables $x_{ijt}$ ($t = 2, \ldots, T$) using equation (3), and removing equations (2) for $t = 2, \ldots, T$, the dual of (LP) can be formulated as

$$\text{maximize} \sum_{j \in \mathcal{S}} v_j + \sum_{t=1}^{T} \sum_{j \in \mathcal{D}} v_{jt} - \sum_{t=1}^{T} \sum_{i=1}^{m} b_{it} \lambda_{it}$$

subject to (D)

$$\begin{array}{rcll} v_j & \leq & \sum_{t=1}^{T} (c_{ijt} + \lambda_{it} d_{jt}) & i = 1, \ldots, m;\ j \in \mathcal{S} \\ v_{jt} & \leq & c_{ijt} + \lambda_{it} d_{jt} & i = 1, \ldots, m;\ j \in \mathcal{D};\ t = 1, \ldots, T \\ \lambda_{i[t+1]} - \lambda_{it} & \leq & h_{it} & i = 1, \ldots, m;\ t = 1, \ldots, T \\ \lambda_{it} & \geq & 0 & i = 1, \ldots, m;\ t = 1, \ldots, T \\ v_j & & \text{free} & j \in \mathcal{S} \\ v_{jt} & & \text{free} & j \in \mathcal{D};\ t = 1, \ldots, T. \end{array}$$

The next result provides some intuition for defining an adequate pseudo-cost function for the greedy heuristic. Moreover, it will also be useful in Section 4 when analyzing the asymptotic feasibility and optimality of our greedy heuristic.

**Proposition 2.2** *Suppose that (LP) is feasible and non-degenerate. Let* $(x^*, I^*)$ *be a basic optimal solution for (LP) and let* $(\lambda^*, v^*)$ *be the corresponding optimal solution for (D). Then,*

*(i) For each* $j \notin B_{\mathcal{S}}$, $x^*_{ijt} = 1$ *for* $t = 1, \ldots, T$ *if and only if*

$$\sum_{t=1}^{T} (c_{ijt} + \lambda^*_{it} d_{jt}) = \min_{l=1,\ldots,m} \sum_{t=1}^{T} (c_{ljt} + \lambda^*_{lt} d_{jt})$$

*and*

$$\sum_{t=1}^{T} (c_{ijt} + \lambda^*_{it} d_{jt}) < \min_{l=1,\ldots,m;\, l \neq i} \sum_{t=1}^{T} (c_{ljt} + \lambda^*_{lt} d_{jt}).$$

*(ii) For each* $j \in B_{\mathcal{S}}$, *there exists an index* $i$ *such that*

$$\sum_{t=1}^{T} (c_{ijt} + \lambda^*_{it} d_{jt}) = \min_{l=1,\ldots,m;\, l \neq i} \sum_{t=1}^{T} (c_{ljt} + \lambda^*_{lt} d_{jt}).$$

*(iii) For each* $(j, t) \notin B_{\mathcal{D}}$, $x^*_{ijt} = 1$ *if and only if*

$$c_{ijt} + \lambda^*_{it} d_{jt} = \min_{l=1,\ldots,m} (c_{ljt} + \lambda^*_{lt} d_{jt})$$

*and*

$$c_{ijt} + \lambda^*_{it} d_{jt} < \min_{l=1,\ldots,m;\, l \neq i} (c_{ljt} + \lambda^*_{lt} d_{jt}).$$

*(iv) For each $(j,t) \in B_{\mathcal{D}}$, there exists an index $i$ such that*

$$c_{ijt} + \lambda^*_{it} d_{jt} = \min_{l=1,\ldots,m;\, l\neq i} (c_{ljt} + \lambda^*_{lt} d_{jt}).$$

**Proof:** See the appendix. □

In the following section, we will show the relationship of (P) with the GAP by reformulating it as a pure assignment problem with nonlinear cost function.

## 2.3 A pure assignment formulation

The original formulation of (P) has assignment variables $x_{ijt}$, as well as inventory level variables $I_{it}$. (P) can be reformulated by replacing the inventory level variables by a nonlinear expression in the assignment variables. The advantage of this is that the problem can be viewed as a pure assignment problem with nonlinear objective function, and that a vector of assignments only can be used to characterize a solution to the problem. This assignment reformulation suggests a class of greedy heuristics for (P) based on the class introduced by Martello and Toth [8] for the GAP, see Section 4 for further details on this heuristic. To reformulate the problem, we define the function $H(x)$ to be the optimal value to the following problem:

$$\text{minimize} \sum_{t=1}^{T} \sum_{i=1}^{m} h_{it} I_{it}$$

subject to

$$\begin{aligned} I_{it} - I_{i[t-1]} &\leq b_{it} - \sum_{j=1}^{n} d_{jt} x_{ijt} && i = 1,\ldots,m;\ t = 1,\ldots,T \\ I_{it} &\geq 0 && i = 1,\ldots,m;\ t = 1,\ldots,T. \end{aligned}$$

We then have the following result:

**Theorem 2.3** *Problem (P) can be equivalently formulated as:*

$$\textit{minimize} \sum_{t=1}^{T} \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ijt} x_{ijt} + H(x)$$

*subject to* *(P′)*

$$\begin{aligned} \sum_{t=1}^{T} \sum_{j=1}^{n} d_{jt} x_{ijt} &\leq \sum_{t=1}^{T} b_{it} && i = 1,\ldots,m \\ \sum_{i=1}^{m} x_{ijt} &= 1 && j = 1,\ldots,n;\ t = 1,\ldots,T \\ x_{ijt} &= x_{ij1} && i = 1,\ldots,m;\ j \in \mathcal{S};\ t = 2,\ldots,T \\ x_{ijt} &\in \{0,1\} && i = 1,\ldots,m;\ j = 1,\ldots,n;\ t = 1,\ldots,T. \end{aligned}$$

**Proof:** Let $F$ be the feasible region of (P). By decomposing (P), we obtain the following equality

$$\begin{aligned}
&\min_{(x,I)\in F}\left(\sum_{t=1}^{T}\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ijt}x_{ijt} + \sum_{t=1}^{T}\sum_{i=1}^{m} h_{it}I_{it}\right) = \\
&= \min_{x:\exists I'\,(x,I')\in F}\left(\sum_{t=1}^{T}\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ijt}x_{ijt} + \min_{I:(x,I)\in F}\sum_{t=1}^{T}\sum_{i=1}^{m} h_{it}I_{it}\right) \\
&= \min_{x:\exists I'\,(x,I')\in F}\left(\sum_{t=1}^{T}\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ijt}x_{ijt} + H(x)\right).
\end{aligned}$$

It remains to be shown that the feasible region of the decomposed problem is equal to the feasible region of (P′). Consider some $x$ so that there exists a feasible solution $(x, I)$ to problem (P). For each facility $i$, we aggregate the capacity constraints over all the periods. Then, we obtain

$$\sum_{t=1}^{T}\left(\sum_{j=1}^{n} d_{jt}x_{ijt} + I_{it}\right) \leq \sum_{t=1}^{T}(b_{it} + I_{i[t-1]})$$

which is equivalent to

$$\sum_{t=1}^{T}\sum_{j=1}^{n} d_{jt}x_{ijt} \leq \sum_{t=1}^{T} b_{it}.$$

This shows that $x$ is feasible for (P′). Now, consider a feasible solution $x$ to (P′). Then, we know there exists a vector $y \in \mathbb{R}^{mT}$ so that

$$y_{it} \leq b_{it} \qquad i = 1, \ldots, m; t = 1, \ldots, T$$

and

$$\sum_{t=1}^{T} y_{it} = \sum_{t=1}^{T}\sum_{j=1}^{n} d_{jt}x_{ijt} \qquad i = 1, \ldots, m$$

(Note that $y$ can be interpreted as a set of feasible production levels corresponding to $(x, I)$ in the original three-level formulation ($P_0$) of the MPSSP.) Now, define $I_{it}$ as

$$I_{it} = \left(\sum_{\tau=1}^{t} y_{i\tau} - \sum_{\tau=1}^{t}\sum_{j=1}^{n} d_{j\tau}x_{ij\tau}\right) - \min_{s=1,\ldots,T}\left(\sum_{\tau=1}^{s} y_{i\tau} - \sum_{\tau=1}^{s}\sum_{j=1}^{n} d_{j\tau}x_{ij\tau}\right)$$

for each $i = 1, \ldots, m$ and $t = 1, \ldots, T$. It is easy to see that $I_{it}$ is nonnegative, and $(x, I) \in F$. This means that $x$ is a feasible solution for the decomposed problem. □

As a final remark, observe that the feasible region of (P′) is equal to that of a GAP.

# 3 A probabilistic analysis of the MPSSP

Consider the following probabilistic model for the parameters of (P)[1]. For each $j = 1,\ldots,n$, let $(D_j, C_j, \Gamma_j)$ be i.i.d. random vectors in $[\underline{D}, \overline{D}]^T \times [\underline{C}, \overline{C}]^{mT} \times \{0,1\}$ (with $\underline{D} > 0$), where $D_j = (D_{jt})_{t=1,\ldots,T}$, $C_j = (C_{ijt})_{i=1,\ldots,m;\, t=1,\ldots,T}$, and $\Gamma_j$ is Bernoulli-distributed with parameter $\pi \in [0,1]$, i.e., $\Gamma_j \sim \text{Be}(\pi)$. We then define

$$\begin{aligned} \mathcal{S} &\equiv \{j : \Gamma_j = 0\} \\ \mathcal{D} &\equiv \{j : \Gamma_j = 1\}. \end{aligned}$$

We assume that the vectors $(D_j, C_j)$ $(j = 1,\ldots,n)$ are i.i.d. according to an absolutely continuous probability distribution for each $j = 1,\ldots,n$. Note that the demands and costs are allowed to be correlated, as are the demands and costs among different periods. The assumption that the vectors of demands and costs are identically distributed for all customers seems restrictive. However, note that this model does allow for a number of classes of customers, each with their own probability distributions, as well as an associated probability of occurrence – handled analogously to the distinction between static and dynamic customers, see above. To allow for sufficient capacity as the number of customers grows, we let $b_{it}$ depend linearly on $n$, i.e., $b_{it} = \beta_{it} n$, for positive constants $\beta_{it}$. This way of modeling the capacities is customary in probabilistic models for assignment problems, see Dyer and Frieze [5], Trick [16], and Romeijn and Piersma [10]. Finally, let $\overline{h} = \max_{i=1,\ldots,m;\, t=1,\ldots,T} h_{it}$, and $\underline{h} = \min_{i=1,\ldots,m;\, t=1,\ldots,T} h_{it}$. Observe that $m$ and $T$ are fixed, thus the size of (P) only depends on the number of customers $n$. In Romeijn and Romero Morales [13] a probabilistic feasibility and value analysis of a variant of the MPSSP with only dynamic customers is performed. Some of the results in this section are generalizations of the results in that paper.

As the following lemma shows, for the probabilistic model described above we have that instances of (LP) are non-degenerate with probability one.

**Lemma 3.1** *(LP) is non-degenerate with probability one, under the stochastic model proposed.*

**Proof:** This follows directly from the fact that the cost and requirement parameters are absolutely continuous random variables. □

As shown by Romeijn and Romero Morales for the GAP [11], feasibility of the problem instances of (P) is not guaranteed under the above stochastic model, even for the LP-relaxation of (P). The following assumption ensures feasibility of (P) with probability one as $n$ goes to infinity.

**Theorem 3.2** *As $n$ goes to infinity, (P) is feasible with probability one if*

$$\sum_{t=1}^{T} \mathcal{E}(D_{1t}) < \sum_{t=1}^{T} \sum_{i=1}^{m} \beta_{it},$$

[1]Throughout the remainder of this paper, random variables will be denoted by capital letters, and their realizations by the corresponding lowercase letters. In addition, the symbol $\mathcal{E}$ will be used to denote *expectation*.

*and infeasible with probability one if the last inequality is reversed.*

**Proof:** This result is a generalization of a special case of the combined results of Theorems 3.2 and 3.5 in Romeijn and Romero Morales [13], and Theorem 2.2 in Romeijn and Romero Morales [12]. The proof is similar to the combined proofs of these results. □

Note that the condition in Theorem 3.2, which is a generalization of Theorem 3.5 in Romeijn and Romero Morales [13], is intuitively appealing since (LP) is feasible if and only if

$$\sum_{j=1}^{n}\sum_{t=1}^{T} d_{jt} \leq \sum_{t=1}^{T}\sum_{i=1}^{m} b_{it}. \tag{6}$$

Apparently it is sufficient to assume a strict inequality version of (6) *in expectation* to be able to conclude *asymptotic* feasibility with probability one for both (LP) and (P). We would like to remark that a similar feasibility result can be derived for acyclic models when all customers are dynamic, see Romeijn and Romero Morales [13]. In the remainder of this paper, we will ensure asymptotic feasibility by explicitly making this assumption:

**Assumption 3.3** *Assume that*

$$\sum_{t=1}^{T} \mathcal{E}(D_{1t}) < \sum_{t=1}^{T}\sum_{i=1}^{m} \beta_{it}.$$

Let $Z_n$ be the random variable representing the optimal value of (P), $Z_n^{\rm LP}$ the optimal value of (LP), and $X^{\rm LP}$ the optimal solution of (LP) (where we have dropped dependence on $n$ in $X^{\rm LP}$ for notational convenience). The following lemma shows that an appropriate normalization of the optimal value of (LP) converges almost surely to a constant.

**Lemma 3.4** *Under Assumption 3.3, there exist constants $\ell$ and $R$ such that, for every $n \geq 1$ and $\delta > 0$,*

$$\Pr\left(\left|\tfrac{1}{n} Z_n^{\rm LP} - \theta\right| \xi_F > \delta\right) \leq \left(\frac{K\delta\sqrt{n}}{\ell R}\right)^{\ell} \cdot \exp\left(-\frac{2\delta^2 n}{R^2}\right)$$

*where $K$ is a universal constant, $\xi_F$ is the indicator function taking the value 1 if the instance is feasible, and 0 otherwise, and where $\theta$ is equal to*

$$\max_{\lambda \in \hat{S}} \left( \pi \mathcal{E}\left( \min_{i=1,\ldots,m} \sum_{t=1}^{T} (C_{i1t} + \lambda_{it} D_{1t}) \right) + (1-\pi) \sum_{t=1}^{T} \mathcal{E}\left( \min_{i=1,\ldots,m} (C_{i1t} + \lambda_{it} D_{1t}) \right) - \lambda^\top \beta \right)$$

*and*

$$\hat{S} = \{\lambda \in \mathbb{R}_+^{mT} : \lambda_{i[t+1]} \leq h_{it} + \lambda_{it},\ i = 1,\ldots,m,\ t = 1,\ldots,T\}.$$

**Proof:** The proof of this result is similar to the proof of Theorem 4.1 in Romeijn and Romero Morales [13]. □

**Corollary 3.5** *Under Assumption 3.3, we have that*

$$\lim_{n\to\infty} \tfrac{1}{n} Z_n^{\mathrm{LP}} = \theta \qquad \textit{almost surely.}$$

**Proof:** See, for instance, Talagrand [15]. □

The intuition behind the expression for $\theta$ is that, if we replace the expectations in the expression of $\theta$ by their finite sample estimators, we obtain, for each realization of assignment costs and demands, the optimal solution value of the dual of the LP-relaxation of the MPSSP, and thus the optimal solution value of the LP-relaxation itself, scaled by a factor of $n$. To be able to show that the appropriately normalized optimal value of (P) converges almost surely to the same constant, we first show that, as $n$ grows to infinity, the aggregate slack in the optimal LP-solution grows linearly in $n$.

**Lemma 3.6** *Under Assumption 3.3,*

$$\sum_{t=1}^{T}\sum_{i=1}^{m} \beta_{it} - \frac{1}{n}\sum_{t=1}^{T}\sum_{i=1}^{m}\sum_{j=1}^{n} D_{jt} X_{ijt}^{\mathrm{LP}} > 0$$

*with probability one when $n$ goes to infinity.*

**Proof:** Note that

$$\begin{aligned}
\sum_{t=1}^{T}\sum_{i=1}^{m} \beta_{it} - \frac{1}{n}\sum_{t=1}^{T}\sum_{i=1}^{m}\sum_{j=1}^{n} d_{jt} x_{ijt}^{\mathrm{LP}} &= \sum_{t=1}^{T}\sum_{i=1}^{m} \beta_{it} - \frac{1}{n}\sum_{t=1}^{T}\sum_{j=1}^{n} d_{jt}\left(\sum_{i=1}^{m} x_{ijt}^{\mathrm{LP}}\right) \\
&= \sum_{t=1}^{T}\sum_{i=1}^{m} \beta_{it} - \frac{1}{n}\sum_{t=1}^{T}\sum_{j=1}^{n} d_{jt} \\
&\to \sum_{t=1}^{T}\sum_{i=1}^{m} \beta_{it} - \sum_{t=1}^{T} \mathcal{E}(D_{1t})
\end{aligned}$$

with probability one as $n$ goes to infinity. Thus, the desired result follows. □

**Theorem 3.7** *Under Assumption 3.3, the following statements hold:*

*(i) With probability one*

$$Z_n \le Z_n^{\mathrm{LP}} + \left((\overline{C} - \underline{C}) + \overline{D} \cdot (\overline{h} - \underline{h}) \cdot (T-1)\right) \cdot mT \tag{7}$$

*as $n$ goes to infinity.*

*(ii) With probability one, $\frac{1}{n} Z_n$ tends to $\theta$ as $n$ goes to infinity.*

**Proof:** We first show that if Claim (i) is true, then Claim (ii) follows. If inequality (7) holds, then

$$\begin{aligned}\left|\tfrac{1}{n}Z_n-\theta\right| &= \left|\left(\tfrac{1}{n}Z_n-\tfrac{1}{n}Z_n^{\mathrm{LP}}\right)+\left(\tfrac{1}{n}Z_n^{\mathrm{LP}}-\theta\right)\right| \\ &\le \left|\tfrac{1}{n}Z_n-\tfrac{1}{n}Z_n^{\mathrm{LP}}\right|+\left|\tfrac{1}{n}Z_n^{\mathrm{LP}}-\theta\right| \\ &\le \tfrac{1}{n}\left((\overline{C}-\underline{C})+\overline{D}\cdot(\overline{h}-\underline{h})\cdot(T-1)\right)\cdot mT+\left|\tfrac{1}{n}Z_n^{\mathrm{LP}}-\theta\right|\end{aligned}$$

with probability one as $n$ goes to infinity, and thus $\frac{1}{n}Z_n$ tends to $\theta$ with probability one as $n$ goes to infinity by using Lemma 3.4. This shows Claim (ii). To complete the proof it remains to show that Claim (i) holds.

The proof of Claim (i) is a generalization of Theorem 4.2 in Romeijn and Romero Morales [13]. Lemma 2.1 says that the number of infeasible assignments in (LP) is at most $mT$. In the following we show that, under Assumption 3.3, we can construct a feasible solution to (P) that only differs from $X^{\mathrm{LP}}$ where the assignment in the latter is infeasible. The difference in objective function values of these two solutions is at most

$$\left((\overline{C}-\underline{C})+\overline{D}\cdot(\overline{h}-\underline{h})\cdot(T-1)\right)\cdot mT,$$

and thus Claim (i) follows.

It remains to show that under Assumption 3.3, there exists a feasible solution to (P) that only differs from $X^{\mathrm{LP}}$ where the assignment in the latter is infeasible. Using Assumption 3.3 and Lemma 3.6, we know that

$$\sum_{t=1}^{T}\sum_{i=1}^{m} b_{it}-\sum_{t=1}^{T}\sum_{i=1}^{m}\sum_{j=1}^{n} d_{jt}X_{ijt}^{\mathrm{LP}} > 2\overline{D}mT^2. \tag{8}$$

We may observe that Lemma 3.6 ensures that as $n$ goes to infinity, the left hand side of this inequality grows linearly in $n$ with probability one, and thus inequality (8) is satisfied with probability one if Assumption 3.3 holds. It is easy to see that inequality (8) implies that

$$\sum_{i=1}^{m}\left\lfloor \sum_{t=1}^{T}\left(b_{it}-\sum_{j=1}^{n} d_{jt}X_{ijt}^{\mathrm{LP}}\right)/(T\overline{D})\right\rfloor \ge mT \tag{9}$$

where $\lfloor\cdot\rfloor$ denotes the largest integer that is no larger than $\cdot$. After fixing the feasible assignments of $X^{\mathrm{LP}}$, the remaining capacity at facility $i$ ($i=1,\ldots,m$) is no less than

$$\sum_{t=1}^{T}\left(b_{it}-\sum_{j=1}^{n} d_{jt}X_{ijt}^{\mathrm{LP}}\right).$$

Since the maximum demand of any assignment is bounded from above by $T\overline{D}$, the infeasible assignments of $X^{\mathrm{LP}}$ can be feasibly accommodated if

$$\sum_{i=1}^{m}\left\lfloor \sum_{t=1}^{T}\left(b_{it}-\sum_{j=1}^{n} d_{jt}X_{ijt}^{\mathrm{LP}}\right)/(T\overline{D})\right\rfloor \ge |B_{\mathcal{S}}|+|B_{\mathcal{D}}|.$$

This inequality holds by using (9) and Lemma 2.1. Therefore, a feasible solution to (P) can be constructed that only differs from $X^{\mathrm{LP}}$ where the assignment in the latter is infeasible, which yields the desired result. □

# 4 An asymptotically optimal greedy heuristic

## 4.1 A class of greedy heuristics

The class of greedy heuristics we propose in this section is similar in spirit to the ones proposed by Martello and Toth [8] for the GAP. (Recall from Section 2.3 that the MPSSP can be formulated as a pure assignment problem.) The idea is that each possible assignment is evaluated by a pseudo-cost function $f(i,j,t)$. For each assignment to be made, the difference between the second smallest and the smallest values of $f(i,j,t)$ (called the *desirability* of making the cheapest assignment with respect to the pseudo-cost) is computed, and assignments are made in decreasing order of this difference. Along the way, the remaining capacities of the facilities, and consequently the values of the desirabilities, are updated to maintain feasibility. Note, from formulation (P′) of the MPSSP, that only the aggregate capacities over time of the facilities are relevant with respect to feasibility.

**Greedy heuristic**

**Step 0.** Set $L = \{1,\ldots,n\} \times \{1,\ldots,T\}$, $B_i = \sum_{t=1}^{T} b_{it}$ for $i = 1,\ldots,m$, and $x^{\mathrm{G}} = 0$.

**Step 1.** For all $(j,t) \in L$, let

$$\begin{aligned} \mathcal{F}_{jt} &= \{i : \sum_{\tau=1}^{T} d_{j\tau} \le B_i\} \text{ for } (j,t) \in L \cap (\mathcal{S} \times \{1,\ldots,T\}) \\ \mathcal{F}_{jt} &= \{i : d_{jt} \le B_i\} \text{ for } (j,t) \in L \cap (\mathcal{D} \times \{1,\ldots,T\}). \end{aligned}$$

If $\mathcal{F}_{jt} = \emptyset$ for some $(j,t) \in L$: let $L = L \backslash \{(j,t)\}$ and repeat Step 1. Otherwise, let

$$\begin{aligned} i_{jt} &\in \arg\min_{i \in \mathcal{F}_{jt}} f(i,j,t) && \text{for } (j,t) \in L \\ \rho_{jt} &= \min_{\substack{s \in \mathcal{F}_{jt} \\ s \neq i_{jt}}} f(s,j,t) - f(i_{jt},j,t) && \text{for } (j,t) \in L. \end{aligned}$$

**Step 2.** Let $(\hat{\jmath},\hat{t}) \in \arg\max_{(j,t)\in L} \rho_{jt}$. If $\hat{\jmath} \in \mathcal{D}$, set

$$\begin{aligned} x^{\mathrm{G}}_{i_{\hat{\jmath}\hat{t}}\hat{\jmath}\hat{t}} &= 1 \\ L &= L \setminus \{(\hat{\jmath},\hat{t})\} \\ B_{i_{\hat{\jmath}\hat{t}}} &= B_{i_{\hat{\jmath}\hat{t}}} - d_{\hat{\jmath}\hat{t}}, \end{aligned}$$

and if $\hat{\jmath} \in \mathcal{S}$, set

$$\begin{aligned} x^{\mathrm{G}}_{i_{\hat{\jmath}\hat{t}}\hat{\jmath}t} &= 1 && \text{for } t = 1,\ldots,T \\ L &= L \setminus \{(\hat{\jmath},t) : t = 1,\ldots,T\} \\ B_{i_{\hat{\jmath}\hat{t}}} &= B_{i_{\hat{\jmath}\hat{t}}} - \sum_{t=1}^{T} d_{\hat{\jmath}t}. \end{aligned}$$

**Step 3.** If $L = \emptyset$: STOP, $x^{\mathrm{G}}$ is a (partial) solution to (P′). Otherwise, go to Step 1.

The output of this greedy heuristic is a vector of assignments $x^{\mathrm{G}}$, which is a *partial* solution of the reformulated problem (P′). Note that Step 2 of the greedy heuristic explicitly ensures that customers in $\mathcal{S}$ are assigned to the same facility in each period. Since, for static customers $j \in \mathcal{S}$, $\mathcal{F}_{jt}$ (in Step 1) is independent of $t$, the demand of that customer in all periods is taken into account when determining the most desirable facility for that customer.

In the following section we show that, for a particular choice of the pseudo-cost function, this greedy heuristic is asymptotically feasible and optimal in a probabilistic sense.

## 4.2 Asymptotic optimality of a greedy heuristic

Following Romeijn and Romero Morales [11], and motivated by Proposition 2.2, we consider the pseudo-cost function given by

$$f(i,j,t) = \begin{cases} \sum_{\tau=1}^{T}(c_{ij\tau} + \lambda^*_{i\tau} d_{j\tau}) & \text{if } j \in \mathcal{S} \\ c_{ijt} + \lambda^*_{it} d_{jt} & \text{if } j \in \mathcal{D} \end{cases}$$

where $\lambda^*$ represents the optimal subvector to (D) corresponding to the capacity constraints of (LP). Theorem 4.2 shows that the (partial) solution found by the greedy heuristic using this pseudo-cost function and the optimal solution for (LP) coincide for almost all assignments that are feasible in the latter. Throughout this section, let $x^{\mathrm{G}}$ denote the (partial) solution for (P′) given by the greedy heuristic, and $z^{\mathrm{G}}$ be its objective value. Let $N$ be the set of assignments which do not coincide in $x^{\mathrm{G}}$ and in $x^{\mathrm{LP}}$, i.e.,

$$N = \{j \in \mathcal{S} : \exists\, i = 1,\ldots,m\; x^{\mathrm{G}}_{ij1} \neq x^{\mathrm{LP}}_{ij1}\} \cup \{(j,t) : j \in \mathcal{D},\, \exists\, i = 1,\ldots,m\; x^{\mathrm{G}}_{ijt} \neq x^{\mathrm{LP}}_{ijt}\}.$$

(Note that, for static customers, we only count the assignment made in period 1, since the assignments in the other periods are necessarily equal to that assignment.) Basically, we will prove that the cardinality of set $N$ is bounded by an expression independent of $n$, and thus the solutions (and, more importantly, their values) stay close when $n$ grows. For reasons of clarity, we will first prove the result when all customers are static (i.e., $\mathcal{D} = \emptyset$) and subsequently prove the result for the general case.

**Theorem 4.1** *Let $\mathcal{D} = \emptyset$. There exists some constant $R_{\mathcal{S}}$, independent of $n$, so that $|N| \leq R_{\mathcal{S}}$ for all instances of (LP) that are feasible and non-degenerate.*

**Proof:** Note that, in this case,

$$N = \{j = 1,\ldots,n : \exists\, i = 1,\ldots,m\; x^{\mathrm{G}}_{ij1} \neq x^{\mathrm{LP}}_{ij1}\}.$$

Clearly, it would be possible to fix all feasible assignments from $x^{\mathrm{LP}}$ without violating any capacity constraints. Proposition 2.2 ensures that the most desirable facility for each customer that is feasibly assigned in $x^{\mathrm{LP}}$ is equal to the facility to which it is assigned in $x^{\mathrm{LP}}$. Moreover, the same proposition shows that the initial desirabilities are such that the greedy heuristic starts by assigning customers that are feasibly assigned in $x^{\mathrm{LP}}$. Now suppose that the greedy heuristic would reproduce all the assignments that are feasible in $x^{\mathrm{LP}}$. Then, because the remaining assignments in $x^{\mathrm{LP}}$ are infeasible with respect to the integrality constraints, $x^{\mathrm{G}}$ and $x^{\mathrm{LP}}$ would differ only in those last ones. By Lemma 2.1 we know that then $|N| \leq mT$, and the result follows. So in the remainder of the proof we will assume that $x^{\mathrm{G}}$ and $x^{\mathrm{LP}}$ differ in at least one assignment that is feasible in the latter.

While the greedy heuristic is assigning customers that are feasibly assigned in $x^{\mathrm{LP}}$ it may at some point start updating the desirabilities of the assignments still to be made due to the decreasing remaining available capacities. This may cause the greedy heuristic to assign a customer that is feasibly assigned in $x^{\mathrm{LP}}$ differently from $x^{\mathrm{LP}}$, and thus the heuristic would deviate from one of the feasible assignments in $x^{\mathrm{LP}}$. Such an assignment could use some available capacity (at most $T\overline{D}$) that $x^{\mathrm{LP}}$ uses for other (feasible) assignments. Since the facility that is involved in this assignment may now not be able to accommodate all customers that were feasibly assigned to it in $x^{\mathrm{LP}}$, other deviations from the feasible assignments in $x^{\mathrm{LP}}$ will occur. However, the number of these deviations is at most equal to $\lceil (T\overline{D})/(T\underline{D}) \rceil = \lceil \overline{D}/\underline{D} \rceil$. In the remainder of this proof we will show that the total number of deviations is bounded by a constant independent of $n$. In order to make this precise, we will first bound the number of times that the desirabilities $\rho$ must be recalculated, and then bound the number of deviations from $x^{\mathrm{LP}}$ between these recalculations.

As mentioned above, we will first bound the number of times that the desirabilities $\rho$ must be recalculated. The calculation of the values of $\rho$ depends only on the set of feasible facilities for each $(j,t) \in L$. The feasibility of a facility is an issue only when its aggregate available capacity is below $T\overline{D}$, and thus the values of $\rho$ only need to be recalculated when, after making an assignment, the aggregate capacity of the corresponding facility is below $T\overline{D}$. Since the aggregate demand for each customer is at least $T\underline{D}$, this happens at most $\lceil \overline{D}/\underline{D} \rceil$ for each facility, and thus the number of times that the desirabilities $\rho$ must be recalculated is no more than $m \lceil \overline{D}/\underline{D} \rceil$.

Now let $l^{(k)}$ be the iteration that induces the $k$-th recalculation of the values of the desirabilities $\rho$, and assume that this recalculation has taken place. Let $M^k$ be the set of customers that have been assigned in the first $l^{(k)}$ iterations, but to a different facility than in $x^{\mathrm{LP}}$. Let $U^k$ be the set of customers that have not been assigned in the first $l^{(k)}$ iterations and for which we would get a different assignment than in $x^{\mathrm{LP}}$ by assigning them to their current (i.e., after the recalculation induced by iteration $l^{(k)}$) most desirable facility (thus, if $j \in U^k$ then $x^{\mathrm{LP}}_{i_j j 1} \neq 1$). In other words, $U^k$ contains the customers that have not been assigned in the first $l^{(k)}$ iterations, and that would belong to $N$ if they were assigned to their most desirable facility.

First note that Proposition 2.2 ensures that initially the most desirable facility in our greedy heuristic for each $j \notin B_{\mathcal{S}}$ coincides with the corresponding assignment in $x^{\mathrm{LP}}$. Moreover, in the original ordering of the desirabilities, we first encounter all customers not in $B_{\mathcal{S}}$, followed by all customers in $B_{\mathcal{S}}$. Since $x^{\mathrm{G}}$ and $x^{\mathrm{LP}}$ do not coincide for at least one customer that is feasibly assigned in $x^{\mathrm{LP}}$, $|M^1| = 0$. Now note that the most desirable facility for all customers *not* in $B_{\mathcal{S}}$ that were *not* assigned in the first $l^{(1)}$ iterations will not change (however, their *second* most desirable facility could change, thereby changing their desirability). Thus, the set of customers not assigned in the first $l^{(1)}$ iterations for which the most desirable facility does not coincide with the corresponding assignment in $x^{\mathrm{LP}}$ is a subset of the set of *infeasible assignments* only in $x^{\mathrm{LP}}$, thus

$$|U^1| \leq |B_{\mathcal{S}}| \leq mT.$$

It is easy to see that, for $k \geq 1$, the number of customers that have been assigned in the first $l^{(k+1)}$ iterations and do not coincide with $x^{\mathrm{LP}}$ is at most equal to the number of customers that have been assigned in the first $l^{(k)}$ iterations and do not coincide with $x^{\mathrm{LP}}$, plus the number of customers that would be assigned to a facility not coinciding with $x^{\mathrm{LP}}$ if they were assigned in one of the iterations $l^{(k)}+1, \ldots, l^{(k+1)}$. In other words,

$$|M^{k+1}| \leq |M^k| + |U^k|. \tag{10}$$

Moreover, the assignments made in the last $l^{(k+1)} - l^{(k)}$ iterations that were different from the corresponding assignment in $x^{\mathrm{LP}}$ could each cause additional deviations from $x^{\mathrm{LP}}$. In particular, each of these assignments could cause at most $\lceil \overline{D}/\underline{D} \rceil$ assignments still to be made to deviate from $x^{\mathrm{LP}}$. Thus,

$$\begin{aligned}
|U^{k+1}| &\leq |U^k| + (|M^{k+1}| - |M^k|) \left\lceil \frac{\overline{D}}{\underline{D}} \right\rceil \\
&\leq |U^k| + |M^{k+1}| \left\lceil \frac{\overline{D}}{\underline{D}} \right\rceil \\
&\leq |U^k| + (|M^k| + |U^k|) \left\lceil \frac{\overline{D}}{\underline{D}} \right\rceil \qquad \text{using equation (10)} \\
&\leq (|M^k| + |U^k|) \left( 1 + \left\lceil \frac{\overline{D}}{\underline{D}} \right\rceil \right).
\end{aligned}$$

Using induction, it can now be shown that

$$|M^k| \leq mT \left( 2 + \left\lceil \frac{\overline{D}}{\underline{D}} \right\rceil \right)^{k-2} \tag{11}$$

$$|U^k| \leq mT \left( 2 + \left\lceil \frac{\overline{D}}{\underline{D}} \right\rceil \right)^{k-1} \tag{12}$$

for each $k$.

If the number of times the desirabilities are recalculated is equal to $k^*$, then $N \subseteq M^{k^*} \cup U^{k^*}$, and thus

$$|N| \leq |M^{k^*}| + |U^{k^*}| \leq 2mT \left( 2 + \left\lceil \frac{\overline{D}}{\underline{D}} \right\rceil \right)^{k^*-1}$$

where the last inequality is derived from (11) and (12). The desired result now follows by observing that $k^* \leq m \left\lceil \frac{\overline{D}}{\underline{D}} \right\rceil$. □

The following theorem generalizes Theorem 4.1 to the mixed case with both static and dynamic customers.

**Theorem 4.2** *There exists some constant $R$, independent of $n$, so that $|N| \leq R$ for all instances of (LP) that are feasible and non-degenerate.*

**Proof:** The proof is similar to the proof of Theorem 4.1. Basically, we again need to bound the number of times the desirabilities $\rho$ must be recalculated, and the number of affected assignments by making an assignment that is different from an assignment in $x^{\mathrm{LP}}$.

The feasibility of a facility is an issue only when its aggregate available capacity is below $T\overline{D}$. Then, it is easy to see that $\rho$ only needs to be recalculated when after an assignment the aggregate capacity of the used facility is below $T\overline{D}$. Moreover, a static assignment uses at least $T\underline{D}$ units of capacity, while a dynamic assignment uses at least $\underline{D}$ units of capacity. Thus, $\rho$ must be recalculated at most $m \left\lceil \frac{T\overline{D}}{\underline{D}} \right\rceil$ times.

An assignment in $x^{\mathrm{G}}$ that is different from the corresponding assignment in $x^{\mathrm{LP}}$ uses at most $T\overline{D}$ units of capacity that $x^{\mathrm{LP}}$ uses for other assignments. Since the minimal demand is bounded from below by $\underline{D}$, an upper bound on the number of possible affected assignments is $\lceil T\overline{D}/\underline{D} \rceil$.

The desired result now easily follows in a similar way as in Theorem 4.1. □

In Theorem 4.3, we state that the greedy heuristic given in Section 4.1 is asymptotically feasible with probability one. This proof combines the results of Theorem 4.2, where it is shown that $x^{\mathrm{LP}}$ and $x^{\mathrm{G}}$ coincide for almost all the feasible assignments in the optimal solution for (LP), and Lemma 3.6.

**Theorem 4.3** *Under Assumption 3.3, the greedy heuristic given in Section 4.1 is asymptotically feasible with probability one.*

**Proof:** From Theorem 4.2, we know that the number of assignments that differ between the optimal solution of (LP) and the solution given by the greedy heuristic is bounded by a constant independent of $n$. Moreover, Lemma 3.6 ensures us that the remaining capacity in the optimal solution for (LP) grows linearly with $n$. Thus, when $n$ grows to infinity, there is enough available capacity to make the remaining assignments (similarly to Theorem 3.7). □

In Theorem 4.4, we show that the greedy heuristic is asymptotically optimal with probability one. The proof is similar to the proof of Theorem 4.3.

**Theorem 4.4** *Under Assumption 3.3, the greedy heuristic given in Section 4.1 is asymptotically optimal with probability one.*

**Proof:** From Theorem 4.3 we know that the greedy heuristic is asymptotically feasible with probability one. It thus suffices to show that $|\frac{1}{n}Z_n^{\mathrm{LP}} - \frac{1}{n}Z_n^{\mathrm{G}}| \to 0$. This follows directly from Theorem 4.2. □

# 5 Computational results

## 5.1 Results of the greedy heuristic

In this section we will illustrate the behavior of the greedy heuristic described in Section 4 on a set of randomly generated test-problems. We first generate a set of customers and a set of facilities uniformly in the square $[0, 10]^2$. For each customer $j = 1, \ldots, n$, we then generate a random demand $D_{jt}$ in period $t$ from the uniform distribution on $[5\sigma_t, 25\sigma_t]$, where $\sigma_t$ is a seasonal factor. We have chosen the vector of seasonal factors to be $\sigma = (\frac{1}{2}, \frac{3}{4}, 1, 1, \frac{3}{4}, \frac{1}{2})^\top$. The costs $C_{ijt}$ are assumed to be proportional to demand and distance, i.e., $C_{ijt} = D_{jt} \cdot \text{dist}_{ij}$, where $\text{dist}_{ij}$ denotes the Euclidean distance between facility $i$ and customer $j$. Finally, we generate inventory holding costs $H_{it}$ uniformly from $[10, 30]$.

We have chosen the capacities $b_{it} = \beta n$, where

$$\beta = \delta \cdot \frac{15}{mT} \sum_{t=1}^{T} \sigma_t.$$

To ensure asymptotic feasibility with probability one of the problems generated, we need to choose $\delta > 1$ (see Theorem 3.2). To account for the asymptotic nature of this feasibility guarantee, we have set $\delta = 1.1$ to obtain feasible instances for finite $n$.

Finally, we have fixed the number of facilities at $m = 5$, and the number of periods at $T = 6$. We have generated and solved 25 instances of the problem for three different cases:

- the purely dynamic case, i.e., $\mathcal{D} = \{1, \ldots, n\}$ and $\mathcal{S} = \emptyset$;
- the purely static case, i.e., $\mathcal{S} = \{1, \ldots, n\}$ and $\mathcal{D} = \emptyset$;
- and a mixed case, where the probabilities that a customer is static or dynamic are both equal to $\frac{1}{2}$, i.e., $\mathcal{E}(|\mathcal{D}|) = \mathcal{E}(|\mathcal{S}|) = \frac{1}{2}n$;

for various numbers of customers.

Table 1 shows the average results of the greedy heuristic. All instances of (LP) were solved using CPLEX 6.6 [3], and the times indicated (in all tables) are CPU-seconds on a PC with a 866 MHz Pentium III processor and 128 MB RAM. The error shown is actually an upper bound on the error, as measured by the relative deviation of the integer solution found from the optimal value of the LP-relaxation.

The main effort in performing the greedy heuristic goes into solving (LP). The table clearly shows that the error of the greedy heuristic decreases as the number of customers increases, illustrating the theoretical result of asymptotic optimality of the greedy heuristic. However, for small numbers of customers the error is rather high, and the greedy heuristic sometimes fails to find a feasible solution even though the instance is feasible. We have therefore studied the effect of adding a local search phase to the greedy heuristic.

[2] The greedy heuristic did not find a feasible solution in 5 instances (one of which is an infeasible instance). The average error is taken over the 20 feasible instances.

| $n$ | static time (sec.) | static error (%) | mixed time (sec.) | mixed error (%) | dynamic time (sec.) | dynamic error (%) |
|---|---|---|---|---|---|---|
| 25 | 0.03 | 11.83[2] | 0.00 | 29.19 | 0.00 | 30.92 |
| 50 | 0.06 | 10.46 | 0.06 | 18.21 | 0.06 | 17.56 |
| 100 | 0.13 | 6.96 | 0.14 | 12.32 | 0.11 | 11.47 |
| 150 | 0.21 | 3.55 | 0.26 | 9.24 | 0.19 | 8.28 |
| 200 | 0.33 | 1.75 | 0.40 | 4.77 | 0.38 | 4.36 |
| 250 | 0.46 | 2.19 | 0.53 | 4.62 | 0.46 | 4.04 |
| 300 | 0.58 | 1.33 | 0.71 | 2.93 | 0.60 | 2.88 |

Table 1: Greedy heuristic

## 5.2 Improving the greedy heuristic solution

In case the greedy heuristic fails to find a feasible solution, we attempt to modify the partial solution provided by the greedy heuristic to obtain a feasible solution. The way this is done is by making local interchanges, thereby creating space for customers (or (customer,period)-pairs) that are unassigned. This improvement phase yields a feasible solution for all feasible problem instances for which the greedy heuristic did not find a feasible solution. The only instances for which the greedy heuristic could not find a feasible solution were static instances with 25 customers. Including the feasibility phase, the average time spent by the greedy heuristic is 0.002 seconds, and the average error (not including the one infeasible instance) is 14.19%.

To improve the value of the feasible solution, we perform a sequence of improving local exchanges between pairs of assignments. The order in which these exchanges are considered is either the natural, lexicographic ordering, or an ordering based on the pseudo-cost function used in the greedy heuristic (see Section 4.2). In particular, for the pure static case, pairs of customers $(j, k)$ are considered for interchange in decreasing order of

$$(f(i_j, j, 1) + f(i_k, k, 1)) - (f(i_k, j, 1) + f(i_j, k, 1))$$

where $i_j$ and $i_k$ are the facilities to which customers $j$ and $k$ are assigned in the current solution. We have compared a strategy where all pairs of customers are considered only once for interchange ('limited local search') with a strategy where interchanges are considered until no improving interchange can be found ('full local search'; i.e., until a true local optimum is found). Similar strategies can be derived for the mixed and pure dynamic case, where, for dynamic customers, customers are replaced by (customer,period)-pairs. For the cases where dynamic customers are present, we have compared a strategy where only interchanges between customers assigned in the same period are considered (i.e., interchanges between pairs $(j, t)$ and $(k, t)$), and a strategy where all interchanges are considered (i.e., interchanges between pairs $(j, t)$ and $(k, t')$). The results are shown in Tables 2–6. As in the previous table, *error* is equal to the relative deviation of the integer solution found from the optimal value of the LP-relaxation.

The results show that ordering the interchange candidates in almost all cases yields a decrease in error that is worthwhile given the small additional amount of time spent

| $n$ | limited local search | | | | full local search | | | |
|---|---|---|---|---|---|---|---|---|
| | not ordered | | ordered | | not ordered | | ordered | |
| | time (sec.) | error (%) | time (sec.) | error (%) | time (sec.) | error (%) | time (sec.) | error (%) |
| 25 | 0.00 | 6.42 | 0.04 | 6.95 | 0.04 | 5.74 | 0.03 | 5.30 |
| 50 | 0.05 | 3.23 | 0.06 | 3.20 | 0.08 | 2.67 | 0.08 | 2.65 |
| 100 | 0.16 | 1.80 | 0.16 | 1.54 | 0.21 | 1.32 | 0.22 | 1.22 |
| 150 | 0.27 | 0.79 | 0.28 | 0.73 | 0.40 | 0.63 | 0.44 | 0.55 |
| 200 | 0.44 | 0.47 | 0.45 | 0.42 | 0.66 | 0.38 | 0.70 | 0.33 |
| 250 | 0.63 | 0.41 | 0.66 | 0.38 | 1.07 | 0.32 | 1.12 | 0.27 |
| 300 | 0.81 | 0.24 | 0.85 | 0.24 | 1.29 | 0.20 | 1.49 | 0.18 |

Table 2: Greedy heuristic + improvement phase; static case

| $n$ | limited local search | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | same period | | | | different periods | | | |
| | not ordered | | ordered | | not ordered | | ordered | |
| | time (sec.) | error (%) | time (sec.) | error (%) | time (sec.) | error (%) | time (sec.) | error (%) |
| 25 | 0.03 | 10.04 | 0.04 | 9.80 | 0.05 | 5.40 | 0.06 | 3.94 |
| 50 | 0.06 | 4.91 | 0.07 | 3.53 | 0.11 | 2.75 | 0.11 | 1.81 |
| 100 | 0.21 | 2.35 | 0.22 | 1.72 | 0.42 | 1.58 | 0.47 | 0.98 |
| 150 | 0.40 | 1.52 | 0.41 | 1.04 | 0.90 | 1.00 | 1.02 | 0.49 |
| 200 | 0.64 | 0.70 | 0.67 | 0.39 | 1.53 | 0.47 | 1.83 | 0.21 |
| 250 | 0.90 | 0.71 | 0.97 | 0.39 | 2.36 | 0.46 | 2.87 | 0.25 |
| 300 | 1.24 | 0.41 | 1.36 | 0.24 | 3.34 | 0.31 | 4.11 | 0.14 |

Table 3: Greedy heuristic + improvement phase; mixed case; limited local search

| $n$ | full local search | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | same period | | | | different periods | | | |
| | not ordered | | ordered | | not ordered | | ordered | |
| | time (sec.) | error (%) | time (sec.) | error (%) | time (sec.) | error (%) | time (sec.) | error (%) |
| 25 | 0.05 | 7.82 | 0.05 | 7.13 | 0.06 | 4.11 | 0.07 | 2.67 |
| 50 | 0.10 | 3.84 | 0.11 | 2.62 | 0.30 | 2.14 | 0.30 | 1.32 |
| 100 | 0.33 | 1.98 | 0.36 | 1.27 | 1.31 | 1.26 | 1.44 | 0.75 |
| 150 | 0.73 | 1.24 | 0.84 | 0.68 | 2.74 | 0.90 | 3.13 | 0.35 |
| 200 | 1.18 | 0.61 | 1.43 | 0.27 | 5.25 | 0.40 | 5.87 | 0.16 |
| 250 | 1.84 | 0.62 | 2.28 | 0.27 | 7.17 | 0.40 | 9.85 | 0.18 |
| 300 | 2.62 | 0.36 | 3.09 | 0.18 | 11.47 | 0.26 | 13.89 | 0.11 |

Table 4: Greedy heuristic + improvement phase; mixed case; full local search

| $n$ | limited local search | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | same period | | | | different periods | | | |
| | not ordered | | ordered | | not ordered | | ordered | |
| | time (sec.) | error (%) | time (sec.) | error (%) | time (sec.) | error (%) | time (sec.) | error (%) |
| 25 | 0.01 | 8.23 | 0.04 | 7.37 | 0.05 | 3.72 | 0.07 | 2.92 |
| 50 | 0.09 | 3.08 | 0.08 | 2.65 | 0.22 | 1.88 | 0.26 | 1.61 |
| 100 | 0.26 | 1.70 | 0.30 | 1.29 | 0.88 | 1.00 | 1.04 | 0.76 |
| 150 | 0.54 | 1.09 | 0.59 | 0.76 | 1.97 | 0.61 | 2.45 | 0.50 |
| 200 | 0.98 | 0.46 | 1.10 | 0.31 | 3.43 | 0.26 | 4.41 | 0.20 |
| 250 | 1.48 | 0.48 | 1.67 | 0.36 | 5.72 | 0.28 | 7.21 | 0.20 |
| 300 | 2.00 | 0.28 | 2.36 | 0.22 | 7.85 | 0.17 | 10.49 | 0.11 |

Table 5: Greedy heuristic + improvement phase; dynamic case; limited local search

| $n$ | full local search | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | same period | | | | different periods | | | |
| | not ordered | | ordered | | not ordered | | ordered | |
| | time (sec.) | error (%) | time (sec.) | error (%) | time (sec.) | error (%) | time (sec.) | error (%) |
| 25 | 0.06 | 6.34 | 0.06 | 5.41 | 0.18 | 2.94 | 0.22 | 2.18 |
| 50 | 0.17 | 2.31 | 0.19 | 1.89 | 0.81 | 1.48 | 0.95 | 1.19 |
| 100 | 0.70 | 1.30 | 0.76 | 0.95 | 3.07 | 0.85 | 3.67 | 0.58 |
| 150 | 1.63 | 0.85 | 1.75 | 0.57 | 7.78 | 0.49 | 9.87 | 0.35 |
| 200 | 2.51 | 0.37 | 3.02 | 0.25 | 12.57 | 0.22 | 16.38 | 0.15 |
| 250 | 4.32 | 0.38 | 5.35 | 0.27 | 22.23 | 0.23 | 26.05 | 0.16 |
| 300 | 6.26 | 0.23 | 7.34 | 0.16 | 28.63 | 0.14 | 36.95 | 0.09 |

Table 6: Greedy heuristic + improvement phase; dynamic case; full local search

| $n$ | static | | mixed | | dynamic | |
|---|---|---|---|---|---|---|
| | solved (%) | time (sec.) | solved (%) | time (sec.) | solved (%) | time (sec.) |
| 25 | 100 | 149.84 | 76 | 512.18 | 0 | - |
| 50 | 96 | 80.75 | 0 | - | - | - |
| 100 | 88 | 224.09 | - | - | - | - |
| 150 | 84 | 408.87 | - | - | - | - |
| 200 | 80 | 488.15 | - | - | - | - |
| 250 | 68 | 631.52 | - | - | - | - |
| 300 | 76 | 728.57 | - | - | - | - |

Table 7: MIP solver of CPLEX

(especially when dynamic customers are present). A very significant decrease in error is obtained when interchanges between assignments in different periods are considered, albeit at a much larger cost in terms of time. Finally, finding a truly local optimum is very expensive, but does yield an additional decrease in error, that may be worthwhile for the smaller problem instances.

As a final remark on the performance of the greedy heuristic, recall that the errors given in the tables are actually upper bounds on the errors, obtained by comparing the value of the greedy heuristic solution to the value of the optimal LP-solution. Note, however, that the (relative) integrality gap may be significant for small instances, even though the relative gap converges to zero (with probability one) as the number of customers increases (see Theorem 3.7), so that the actual error in the greedy heuristic solution can be significantly smaller than the error shown.

In order to illustrate the difficulty of the MPSSP, we have used the MIP solver of CPLEX to try to solve the same instances as above. Recall that the greedy heuristic together with the improving phase was able to find a feasible solution in all cases where the instance was feasible, and its objective value was given to the MIP solver of CPLEX as an upper bound. In all cases, this yields a high quality upper bound to the MPSSP, which of course in itself will yield a significant speedup of the CPLEX solver compared to the case where no upper bound is provided. To limit the time needed for the computational experiments, we decided to allow a maximum of 1 hour of CPU time per instance. Within this time limit, the MIP solver of CPLEX often failed to solve the problem satisfactorily, particularly for the mixed and dynamic cases (see in Table 7). The column *solved* indicates the percentage of instances where an optimal solution was found within 1 hour, and *time* is equal to the average time spent on solving these instances. The static instances get harder as the number of customers grows, but most of the instances can be solved satisfactorily (of course using significantly more time than the greedy heuristic). However, instances of the mixed case with just 50 customers cannot be solved within the 1-hour time limit, and neither can purely dynamic instances with only 25 customers, which illustrates the value of the greedy heuristic developed in this paper.

# 6 Summary and future research

In this paper we have considered the problem of evaluating a logistics network design in a dynamic environment. We have proposed a new class of pseudo-cost functions for the greedy heuristic that was developed by Martello and Toth [8] for the Generalized Assignment Problem, and have shown that a particular element from that class yields a greedy heuristic that is asymptotically feasible and optimal in a probabilistic sense. This behavior is illustrated with some numerical results of the greedy heuristic. In addition, it is shown that significant improvements can be made by using the result of the greedy heuristic as the starting point of a local interchange procedure, yielding very nearly optimal solutions for problems with many customers. We are currently investigating the merits of solving smaller instances to optimality using a column generation approach.

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# Appendix

Let $(x^*, I^*)$ be a basic optimal solution for (LP). In the following lemma, which will be used in the proof of Proposition 2.2, we derive a relationship between the number of split customers, the number of fractional assignment variables, the number of times a facility is used to full capacity in a period, and the number of strictly positive inventory variables. Let $F_{\mathcal{S}}$ be the set of fractional assignment variables in $(x^*, I^*)$ associated with customers in $\mathcal{S}$ (where each of these assignments is counted only for period 1, since the values of the assignment variables are equal for all periods) and $F_{\mathcal{D}}$ be the set of fractional assignment variables in $(x^*, I^*)$ associated with customers in $\mathcal{D}$, $M$ be the set of (facility, period)-pairs such that $(i,t) \in M$ means that facility $i$ is used to full capacity period $t$, and $I^+$ be the set of strictly positive inventory variables in $I^*$. These sets can be expressed as

$$\begin{aligned}
F_{\mathcal{S}} &= \{(i,j) : j \in \mathcal{S},\, 0 < x^*_{ij1} < 1\} \\
F_{\mathcal{D}} &= \{(i,j,t) : j \in \mathcal{D},\, 0 < x^*_{ijt} < 1\} \\
M &= \{(i,t) : \sum_{j=1}^{n} d_{jt} x^*_{ijt} + I^*_{it} = b_{it} + I^*_{i[t-1]}\} \\
I^+ &= \{(i,t) : I^*_{it} > 0\}.
\end{aligned}$$

**Lemma A.1** *If (LP) is feasible and non-degenerate, then for a basic optimal solution $(x^*, I^*)$ of (LP) we have that*

$$|F_{\mathcal{S}}| + |F_{\mathcal{D}}| + |I^+| = |M| + |B_{\mathcal{S}}| + |B_{\mathcal{D}}|.$$

**Proof:** Denote by $s_{it}$ the surplus variables corresponding to the capacity constraints in (LP). Thus, including these variables, and eliminating the variables $x_{ijt}$ for $i \in \mathcal{S}$, $j = 1, \ldots, n$, and $t = 2, \ldots, T$, (LP) can be reformulated as

$$\text{minimize } \sum_{t=1}^{T} \sum_{i=1}^{m} \sum_{j \in \mathcal{S}} c_{ijt} x_{ij1} + \sum_{t=1}^{T} \sum_{i=1}^{m} \sum_{j \in \mathcal{D}} c_{ijt} x_{ijt} + \sum_{t=1}^{T} \sum_{i=1}^{m} h_{it} I_{it}$$

subject to

$$
\begin{array}{rcll}
\sum_{j\in\mathcal{S}} d_{jt}x_{ij1} + \sum_{j\in\mathcal{D}} d_{jt}x_{ijt} + I_{it} + s_{it} & = & b_{it} + I_{i[t-1]} & i=1,\ldots,m;\ t=1,\ldots,T \\
\sum_{i=1}^{m} x_{ij1} & = & 1 & j\in\mathcal{S} \\
\sum_{i=1}^{m} x_{ijt} & = & 1 & j\in\mathcal{D};\ t=1,\ldots,T \\
x_{ij1} & \geq & 0 & i=1,\ldots,m;\ j\in\mathcal{S} \\
x_{ijt} & \geq & 0 & i=1,\ldots,m;\ j\in\mathcal{D};\ t=1,\ldots,T \\
I_{it} & \geq & 0 & i=1,\ldots,m;\ t=1,\ldots,T \\
s_{it} & \geq & 0 & i=1,\ldots,m;\ t=1,\ldots,T.
\end{array}
$$

Let $(x^*, I^*, s^*)$ be a basic optimal solution for (LP). Then, the set $M$, defined above, is equal to

$$M = \{(i,t) : s^*_{it} = 0\}.$$

Under non-degeneracy, the number of non-zero variables at $(x^*, I^*, s^*)$ is equal to $mT + |\mathcal{S}| + |\mathcal{D}| \cdot T$, the number of constraints in (LP). The number of non-zero assignment variables is equal to $(|\mathcal{S}| - |B_{\mathcal{S}}|) + |F_{\mathcal{S}}| + (|\mathcal{D}| \cdot T - |B_{\mathcal{D}}|) + |F_{\mathcal{D}}|$, where the first term corresponds to the variables $x^*_{ij1} = 1$ for $j \in \mathcal{S}$, the second one to the fractional assignment variables associated with $j \in \mathcal{S}$, analogously, the third term corresponds to the variables $x^*_{ijt} = 1$ for $j \in \mathcal{D}$, the fourth one to the fractional assignment variables associated with $j \in \mathcal{D}$. With respect to the surplus variables, we have $mT - |M|$ non-zero variables. By definition $|I^+|$ is the number of non-zero inventory variables. Thus, by imposing that the number of non-zero variables at $(x^*, I^*, s^*)$ is equal to $mT + |\mathcal{S}| + |\mathcal{D}| \cdot T$, we obtain

$$mT + |\mathcal{S}| + |\mathcal{D}| \cdot T = (|\mathcal{S}| - |B_{\mathcal{S}}|) + |F_{\mathcal{S}}| + (|\mathcal{D}| \cdot T - |B_{\mathcal{D}}|) + |F_{\mathcal{D}}| + mT - |M| + |I^+|.$$

The desired result now follows from the last equality. □

**Proposition 2.2** *Suppose that (LP) is feasible and non-degenerate. Let $(x^*, I^*)$ be a basic optimal solution for (LP) and let $(\lambda^*, v^*)$ be the corresponding optimal solution for (D). Then,*

*(i) For each $j \notin B_{\mathcal{S}}$, $x^*_{ijt} = 1$ for $t = 1, \ldots, T$ if and only if*

$$\sum_{t=1}^{T} (c_{ijt} + \lambda^*_{it} d_{jt}) = \min_{l=1,\ldots,m} \sum_{t=1}^{T} (c_{ljt} + \lambda^*_{lt} d_{jt})$$

*and*

$$\sum_{t=1}^{T} (c_{ijt} + \lambda^*_{it} d_{jt}) < \min_{l=1,\ldots,m;\, l\neq i} \sum_{t=1}^{T} (c_{ljt} + \lambda^*_{lt} d_{jt}).$$

*(ii) For each $j \in B_{\mathcal{S}}$, there exists an index $i$ such that*

$$\sum_{t=1}^{T}(c_{ijt} + \lambda^*_{it} d_{jt}) = \min_{l=1,\ldots,m;\, l\neq i} \sum_{t=1}^{T}(c_{ljt} + \lambda^*_{lt} d_{jt}).$$

*(iii) For each $(j,t) \notin B_{\mathcal{D}}$, $x^*_{ijt} = 1$ if and only if*

$$c_{ijt} + \lambda^*_{it} d_{jt} = \min_{l=1,\ldots,m} (c_{ljt} + \lambda^*_{lt} d_{jt})$$

*and*

$$c_{ijt} + \lambda^*_{it} d_{jt} < \min_{l=1,\ldots,m;\, l\neq i} (c_{ljt} + \lambda^*_{lt} d_{jt}).$$

*(iv) For each $(j,t) \in B_{\mathcal{D}}$, there exists an index $i$ such that*

$$c_{ijt} + \lambda^*_{it} d_{jt} = \min_{l=1,\ldots,m;\, l\neq i} (c_{ljt} + \lambda^*_{lt} d_{jt}).$$

**Proof:** Observe that

$$\begin{array}{rcll} v^*_j &=& \min_{l=1,\ldots,m} \sum_{t=1}^{T}(c_{ljt} + \lambda^*_{lt} d_{jt}) \geq 0 & \text{for } j \in \mathcal{S} \\ v^*_{jt} &=& \min_{l=1,\ldots,m} (c_{ljt} + \lambda^*_{lt} d_{jt}) \quad \geq 0 & \text{for } j \in \mathcal{D},\ t = 1,\ldots,T. \end{array}$$

Thus, without loss of optimality, we can add to (D) the non-negativity constraints on the vector $v$. By adding surplus variables $s_{ij}$, $s_{ijt}$, and $S_{it}$ to the constraints in (D), we can reformulate it as

$$\text{maximize} \sum_{j\in\mathcal{S}} v_j + \sum_{t=1}^{T}\sum_{j\in\mathcal{D}} v_{jt} - \sum_{t=1}^{T}\sum_{i=1}^{m} b_{it}\lambda_{it}$$

subject to (D′)

$$\begin{array}{rcll} v_j + s_{ij} &=& \sum_{t=1}^{T}(c_{ijt} + \lambda_{it} d_{jt}) & i = 1,\ldots,m;\ j \in \mathcal{S} \\ v_{jt} + s_{ijt} &=& c_{ijt} + \lambda_{it} d_{jt} & i = 1,\ldots,m;\ j \in \mathcal{D};\ t = 1,\ldots,T \\ \lambda_{i[t+1]} - \lambda_{it} + S_{it} &=& h_{it} & i = 1,\ldots,m;\ t = 1,\ldots,T \\ \lambda_{it} &\geq& 0 & i = 1,\ldots,m;\ t = 1,\ldots,T \\ v_j &\geq& 0 & j \in \mathcal{S} \\ v_{jt} &\geq& 0 & j \in \mathcal{D} \\ s_{ij} &\geq& 0 & i = 1,\ldots,m;\ j \in \mathcal{S} \\ s_{ijt} &\geq& 0 & i = 1,\ldots,m;\ j \in \mathcal{D};\ t = 1,\ldots,T \\ S_{it} &\geq& 0 & i = 1,\ldots,m;\ t = 1,\ldots,T-1. \end{array}$$

Let $(\lambda^*, v^*, s^*, S^*)$ be the optimal solution for (D′). For each $j \in B_{\mathcal{S}}$, there exist at least two variables $x^*_{ij1}$ that are strictly positive. Hence, by the complementary slackness conditions, there exist at least two variables $s^*_{ij}$ equal to zero. This proves Claim (ii). The same applies to $(j,t) \in B_{\mathcal{D}}$, thus, Claim (iv) follows.

To prove Claims (i) and (iii), it is enough to show that for each $j \notin B_{\mathcal{S}}$ there exists exactly one variable $s^*_{ij} = 0$ and for each $(j,t) \notin B_{\mathcal{D}}$ there exists exactly one variable $s^*_{ijt} = 0$. By complementary slackness conditions we know that at least there exists one of these variables. We have to show the uniqueness, and we do it by counting the variables at level zero in the vector $(\lambda^*, v^*, s^*, S^*)$. There are at least $mT - |M|$ variables $\lambda^*_{it}$, $|F_{\mathcal{S}}|$ variables $s^*_{ij}$ corresponding to $j \in B_{\mathcal{S}}$, $|F_{\mathcal{D}}|$ variables $s^*_{ijt}$ corresponding to $(j,t) \in B_{\mathcal{D}}$, $|\mathcal{S}| - |B_{\mathcal{S}}|$ variables $s^*_{ij}$ corresponding to $j \notin B_{\mathcal{S}}$, $|\mathcal{D}| \cdot T - |B_{\mathcal{D}}|$ variables $s^*_{ijt}$ corresponding to $(j,t) \notin B_{\mathcal{D}}$, and $|I^+|$ variables $S_{it}$ equal to zero. In total, we have at least $mT - |M| + |F_{\mathcal{S}}| + (|\mathcal{S}| - |B_{\mathcal{S}}|) + |F_{\mathcal{D}}| + (|\mathcal{D}| \cdot T - |B_{\mathcal{D}}|) + |I^+| = mT + |\mathcal{S}| + |\mathcal{D}| \cdot T$ zeroes in the optimal dual solution, where the last equality follows from Lemma A.1. So, these are exactly all the variables at level zero in vector $(\lambda^*, v^*, s^*, S^*)$. Then, for each $j \notin B_{\mathcal{S}}$ there exists exactly one variable $s^*_{ij} = 0$, and Claim (i) follows. Claim (iii) now follows in a similar way. □