Friends or Foes?

The Interrelationship between Angel and Venture Capital Markets

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— ONLINE APPENDIX —

Angel market: equilibrium equity shares and entrepreneur’s outside option.

According to the Nash product, \( \alpha^* \) is implicitly defined by

\[
\frac{dD_E(e_1^*)}{d\alpha} D_A^+(e_1^*) + (D_E^+(e_1^*) - U_E^1) \frac{dD_A^+(e_1^*)}{d\alpha} = 0.
\]  

(A.1)

Applying the Envelope Theorem we find that \( \frac{dD_E(e_1^*)}{d\alpha} < 0 \). We can then infer from Eq. (A.1) that \( \frac{dD_A(e_1^*)}{d\alpha} > 0 \) must hold for \( \alpha = \alpha^* \). Using Eq. (A.1) we can implicitly differentiate \( \alpha^* \) w.r.t. \( U_E^1 \):

\[
\frac{d\alpha^*}{dU_E^1} = \frac{dD_A^+(e_1^*)}{d\alpha} \frac{dD_A^+(e_1^*)}{d\alpha} + (D_E^+(e_1^*) - U_E^1) \frac{dD_A^+(e_1^*)}{d\alpha}.
\]  

(A.2)

Note that the denominator is strictly negative due to the second-order condition for \( \alpha^* \). Moreover, recall that \( \frac{dD_A^+(e_1^*)}{d\alpha} > 0 \). Thus, \( \frac{d\alpha^*}{dU_E^1} < 0 \).

Angel market: optimal transfer payment.

Suppose the angel makes the transfer \( T \) to the entrepreneur in exchange for an additional equity stake \( \tilde{\alpha}(T) \). The angel’s new equity share is then given by \( \alpha(T) \equiv \alpha^* + \tilde{\alpha}(T) \), with \( \alpha'(T) > 0 \), \( \alpha(T) \geq 0 \forall T \geq 0 \), and \( \alpha(T) < 0 \forall T < 0 \). Note that any post bargaining transfers aimed at adjusting the equity allocation, must improve joint efficiency to be implementable.

The joint utility at the deal stage is

\[
D_A^1 + D_E^1 = \rho_1(e_1) \left[ g \left( U_1^A + U_2^E \right) + (1 - g) y_1 \right] - k_1 - c(e_1),
\]

(A.3)

where \( e_1 \equiv e_1(\alpha(T)) \). Thus, the marginal effect of a transfer \( T \) on joint utility is given by

\[
\frac{d \left[ D_A^1 + D_E^1 \right]}{dT} = \left[ \rho'_1(e_1) g \left( U_2^A + U_2^E \right) - c'(e_1) \right] \frac{de_1}{d\alpha(T)} \frac{d\alpha(T)}{dT}.
\]  

(A.4)
Recall that \( \frac{dc_{1}}{d\alpha} < 0 \), so that \( e_{1} \) is maximized at \( \alpha = 0 \). Moreover, \( X \geq 0 \). Thus, \( d[D_{1}^{A} + D_{1}^{E}] /dT > 0 \) requires that \( d\alpha(T)/dT < 0 \), and therefore \( T < 0 \). However, because of his zero wealth, the entrepreneur cannot make a payment to the angel. Thus, \( T^* = 0 \).

**Derivation of angel market equilibrium.**

Using \( q_{1}^{A} = x_{1}/M_{1}^{A} \) and \( x_{1} = \phi_{1} [M_{1}^{E}M_{1}^{A}]^{0.5} \), we can write Eq. (6) as

\[
\phi_{1} D_{1}^{A} \left[ \frac{M_{1}^{E}}{M_{1}^{A}} \right]^{0.5} = \sigma_{1}^{A}.
\] (A.5)

Using \( \theta_{1} = M_{1}^{A}/M_{1}^{E} \) we then get the equilibrium degree of competition for the angel market:

\[
\theta_{1}^{*} = \left( \frac{\phi_{1} D_{1}^{A}/\sigma_{1}^{A}}{\sigma_{1}^{A}} \right)^{2}.
\] (A.6)

Solving Eq. (8) for \( M_{1}^{E} \) and using \( q_{1}^{E} = \phi_{1} [M_{1}^{E}M_{1}^{A}]^{0.5}/M_{1}^{E} = \phi_{1} [M_{1}^{A}/M_{1}^{E}]^{0.5} \), we get the equilibrium stock of entrepreneurs in the early stage market:

\[
M_{1}^{E*} = \frac{F(U_{1}^{E})}{\delta_{1} + q_{1}^{E}} = \frac{F(U_{1}^{E})}{\delta_{1} + \phi_{1} [M_{1}^{A}/M_{1}^{E}]^{0.5}} = \frac{F(U_{1}^{E})}{\delta_{1} + \phi_{1} \sqrt{\theta_{1}^{*}}}. \] (A.7)

Thus, the equilibrium stock of angels is given by

\[
M_{1}^{A*} = M_{1}^{E*} \theta_{1}^{*} = \frac{F(U_{1}^{E})\theta_{1}^{*}}{\delta_{1} + \phi_{1} \sqrt{\theta_{1}^{*}}}. \] (A.8)

Using \( M_{1}^{E*} = M_{1}^{A*}/\theta_{1}^{*} \) we can then write \( x_{1}^{*} \) as

\[
x_{1}^{*} = \phi_{1} [M_{1}^{A*}M_{1}^{E*}]^{0.5} = \frac{\phi_{1} M_{1}^{A*} \sqrt{\theta_{1}^{*}}}{\delta_{1} + \phi_{1} \sqrt{\theta_{1}^{*}}} = \frac{\phi_{1} \sqrt{\theta_{1}^{*}}}{\delta_{1} + \phi_{1} \sqrt{\theta_{1}^{*}}}. \] (A.9)

Moreover, using Eq. (9) and \( q_{1}^{A} = x_{1}/M_{1}^{A} \) we get \( m_{1}^{A*} = q_{1}^{A} M_{1}^{A*} = x_{1}^{*} \).

**Proof of Proposition 1.**

Recall that the equilibrium of the angel market is determined by the deal values \( D_{1}^{E} \) and \( D_{1}^{A} \), and therefore by the late stage utilities \( U_{2}^{E} \) and \( U_{2}^{A} \), as well as by the entrepreneur’s outside option \( U_{1}^{E} \) (through \( \alpha* \)). We will show in Proof of Proposition 4 that \( U_{2}^{E} \) and \( U_{2}^{A} \) do not depend on \( \phi_{1}, \delta_{1}, \sigma_{1}^{E}, \sigma_{1}^{A}, \) and \( k_{1} \). Next we need to derive a condition which defines \( U_{1}^{E} \). The equilibrium condition (5) can be written as

\[
U_{1}^{E} [r + \delta_{1}] = -\sigma_{1}^{E} + q_{1}^{E} [D_{1}^{E} - U_{1}^{E}] . \] (A.10)
Using $q_1^E = \phi_1 \left[ M_1^{A*}/M_1^E \right]^{0.5} = \phi_1 \sqrt{\theta_1^*} = \phi_1^2 D_1^A / \sigma_1^A$ we get the following condition which defines $U_1^E$:

$$U_1^E [r + \delta_1] - \frac{\phi_1^2}{\sigma_1^A} D_1^A \left[ D_1^E - U_1^E \right] + \sigma_1^E = 0. \quad \text{(A.11)}$$

Now consider the equilibrium degree of competition $\theta_1^*$. Differentiating $\theta_1^*$ w.r.t. $\delta_1$ yields

$$\frac{d\theta_1^*}{d\delta_1} = 2 \frac{\phi_1^2 D_1^A}{[\sigma_1^A]^2} \frac{dD_1^A}{d\delta_1} = 2 \frac{\phi_1^2 D_1^A}{[\sigma_1^A]^2} \frac{d\alpha^*}{d\alpha} \frac{dU_1^E}{d\alpha} \frac{dU_1^E}{d\delta_1}. \quad \text{(A.12)}$$

Next we define $\Gamma \equiv D_1^A \left[ D_1^E - U_1^E \right]$. We then get

$$\frac{dU_1^E}{d\delta_1} = - \frac{U_1^E}{r + \delta_1 + \frac{\phi_1^2}{\sigma_1^A} D_1^A}. \quad \text{(A.13)}$$

Note that $d\Gamma/d\alpha = 0$ due to the first-order condition for $\alpha^*$. Moreover, $\partial \Gamma / \partial U_1^E = -D_1^A$. Consequently,

$$\frac{dU_1^E}{d\delta_1} = - \frac{U_1^E}{r + \delta_1 + \frac{\phi_1^2}{\sigma_1^A} D_1^A} < 0. \quad \text{(A.14)}$$

This in turn implies that $d\theta_1^*/d\delta_1 > 0$. Likewise,

$$\frac{d\theta_1^*}{d\sigma_1^E} = 2 \frac{\phi_1^2 D_1^A}{[\sigma_1^A]^2} \frac{d\alpha^*}{d\alpha} \frac{dU_1^E}{d\alpha} \frac{dU_1^E}{d\sigma_1^E}, \quad \text{(A.15)}$$

with $dD_1^A/d\alpha > 0$, $d\alpha^*/dU_1^E < 0$, and

$$\frac{dU_1^E}{d\sigma_1^E} = - \frac{1}{r + \delta_1 + \frac{\phi_1^2}{\sigma_1^A} D_1^A} < 0. \quad \text{(A.16)}$$

Thus, $d\theta_1^*/d\sigma_1^E > 0$. Moreover, note that $dD_1^A/dk_1 < 0$. Consequently, $d\theta_1^*/dk_1 < 0$. For the remaining comparative statics it is useful to express the condition for $U_1^E$ in terms of $\theta_1^*$:

$$U_1^E [r + \delta_1] - \phi_1 \sqrt{\theta_1^*} \left[ D_1^E - U_1^E \right] + \sigma_1^E = 0, \quad \text{(A.17)}$$

so that

$$\frac{dU_1^E}{d\theta_1^*} = \frac{\phi_1}{2 \sqrt{\theta_1^*}} \left[ D_1^E - U_1^E \right] \frac{1}{r + \delta_1 + \phi_1 \sqrt{\theta_1^*}} > 0. \quad \text{(A.18)}$$

Moreover, using the definition of $\theta_1^*$ we define

$$G \equiv \theta_1^* - \left[ \frac{\phi_1}{\sigma_1^A} D_1^A \right]^2 = 0 \quad \text{(A.19)}$$
where $D^A_1 = \alpha^*(U_1^E(\theta_1^*))$. We get

$$\frac{d\theta_1^*}{d\phi_1} = \frac{2\frac{\phi_1^*}{\sigma_1^*} D^A_1}{1 - 2 \left[ \frac{\phi_1^*}{\sigma_1^*} \right]^2 D^A_1 \frac{dD^A_1}{d\sigma_1^*} \frac{dU_1^E}{d\theta_1^*}}. \tag{A.20}$$

Recall that $dD^A_1/d\alpha > 0$, $d\alpha^*/dU_1^E < 0$, and $dU_1^E/d\theta_1^* > 0$. Thus, the denominator is positive, which implies that $d\theta_1^*/d\phi_1 > 0$. Likewise, using Eq. (A.19), we get

$$\frac{d\theta_1^*}{d\sigma_1^*} = \frac{2\frac{\phi_1^*}{\sigma_1^*} D^A_1}{1 - 2 \left[ \frac{\phi_1^*}{\sigma_1^*} \right]^2 D^A_1 \frac{dD^A_1}{d\sigma_1^*} \frac{dU_1^E}{d\theta_1^*}}. \tag{A.21}$$

Again, the denominator is positive, which implies that $d\theta_1^*/d\sigma_1^* < 0$.

Next, note that $dm_1^{E^*}/dU_1^E = F'(U_1^E) > 0$, and recall that $dU_1^E/d\delta_1$, $dU_1^E/d\sigma_1^* < 0$. Moreover, using Eq. (A.11) we find

$$\frac{dU_1^E}{d\phi_1} = \frac{2\frac{\phi_1^*}{\sigma_1^*} D^A_1 \left[ D_1^E - U_1^E \right]}{r + \delta_1 + \frac{\phi_1^*}{\sigma_1^*} D^A_1} > 0 \tag{A.22}$$

$$\frac{dU_1^E}{d\sigma_1^*} = \frac{-\frac{\phi_1^*}{\sigma_1^*} D^A_1 \left[ D_1^E - U_1^E \right]}{r + \delta_1 + \frac{\phi_1^*}{\sigma_1^*} D^A_1} < 0. \tag{A.23}$$

Likewise, using $\Gamma = D^A_1 \left[ D_1^E - U_1^E \right]$,

$$\frac{dU_1^E}{dk_1} = \frac{\frac{\phi_1^*}{\sigma_1^*} \frac{\partial \Gamma}{\partial k_1}}{r + \delta_1 + \frac{\phi_1^*}{\sigma_1^*} D^A_1}, \tag{A.24}$$

with

$$\frac{d\Gamma}{dk_1} = \frac{d\Gamma}{d\alpha} \frac{d\alpha}{dk_1} + \frac{\partial \Gamma}{\partial k_1} = - \left[ D_1^E - U_1^E \right] < 0. \tag{A.25}$$

Thus, $dU_1^E/dk_1 < 0$. All this implies that $m_1^{E^*}$ is increasing in $\phi_1$, and decreasing in $\delta_1$, $\sigma_1^E$, $\sigma_1^A$, and $k_1$.

Next, recall that $m_1^{A^*} = x_1^*$ is given by

$$m_1^{A^*} = x_1^* = F(U_1^E) \frac{\phi_1 \sqrt{\theta_1^*}}{\delta_1 + \phi_1 \sqrt{\theta_1^*}}. \tag{A.26}$$

It is straightforward to show that $dT/d(\phi_1 \sqrt{\theta_1^*}) > 0$. Because $dU_1^E/d\phi_1 > 0$ and $d\theta_1^*/d\phi_1 > 0$, we then have $dm_1^{A^*}/d\phi_1 = dx_1^*/d\phi_1 > 0$. Likewise, we know that $dU_1^E/d\sigma_1^A$, $dU_1^E/dk_1 < 0$
and \(d\theta^*/d\sigma^A, d\theta^*/dk_1 < 0\). Thus, \(dm^A/d\sigma^A = dx_1^*/d\sigma^A < 0\) and \(dm^A*/dk_1 = dx_1^*/dk_1 < 0\). Moreover, we have shown that \(dU^E/d\delta_1, dU^E/d\sigma^E < 0\), while \(d\theta^*/d\delta_1, d\theta^*/d\sigma^E > 0\). Thus, the effects of \(\delta_1\) and \(\sigma^E\) on \(m_i^A = x_1^*\) are ambiguous.

Now consider the equilibrium valuation \(V_1^*\). Note that \(V_1^*\) is decreasing in the angel’s equilibrium equity share \(\alpha^*\), which is defined by Eq. (A.1). Recall that \(d\alpha^*/dU^E < 0\), \(dU^E/d\phi_1 > 0\) and \(dU^E/d\delta_1, dU^E/d\sigma^E, dU^E/d\sigma^A < 0\). Consequently, \(d\alpha^*/d\phi_1 < 0\) and \(d\alpha^*/d\delta_1, d\alpha^*/d\sigma^E, d\alpha^*/d\sigma^A > 0\). All this implies that \(V_1^*\) is increasing in \(\phi_1\), and decreasing in \(\delta_1, \sigma^E\) and \(\sigma^A\). Furthermore, note that \(k_1\) affects \(D_1^A\) and \(U_1^A\). Using Eq. (A.1) we get

\[
\frac{d\alpha^*}{dk_1} = -\frac{dD_1^A}{d\alpha} \frac{d\alpha^*}{dk_1} = \frac{-\frac{dD_1^A}{d\alpha} \frac{d\alpha^*}{dk_1} + (D_1^E - U_1^E) \frac{d\alpha^*}{d\alpha}}{D_1^A + (D_1^E - U_1^E) \frac{d\alpha^*}{d\alpha}},
\]

(A.27)

where the denominator is strictly negative due to the second-order condition for \(\alpha^*\). Thus, to prove that \(d\alpha^*/dk_1 > 0\), we need to show that the numerator is positive. We know that \(dD_1^A/d\alpha < 0, dD_1^A/d\alpha > 0, \) and \(dU_1^E/dk_1 < 0\). Moreover, \(\partial D_1^A/\partial k_1 = -1\) and \(d^2 D_1^A/(d\alpha dk_1) = 0\). Thus, the numerator is strictly positive, so that \(d\alpha^*/dk_1 > 0\). This in turn implies that the effect of \(k_1\) on \(V_1^* = k_1/\alpha^*\) is ambiguous.

Finally consider the equilibrium success probability \(\rho_1(e_1^*)\), with \(\rho_1'(e_1^*) > 0\). Using Eq. (3) we get

\[
\frac{de_1^*}{d\alpha} = \frac{\rho_1'(e_1^*) (1 - q)y_1}{\rho_1'(e_1^*) [qU_2^A + (1 - q)(1 - \alpha)y_1 + c'(e_1)]},
\]

(A.28)

where the denominator is strictly negative due to the second-order condition for \(e_1^*\). Thus, \(de_1^*/d\alpha < 0\). Our comparative statics results for \(\alpha^*\) then imply that \(d\rho_1(e_1^*)/d\phi_1 > 0\) and \(d\rho_1(e_1^*)/d\delta_1, d\rho_1(e_1^*)/d\sigma^E, d\rho_1(e_1^*)/d\sigma^A, d\rho_1(e_1^*)/dk_1 < 0\).

**Early Stage Investment and Valuation.**

Consider first our base model with endogenous effort. Differentiating \(V_1^*\) w.r.t. \(k_1\) yields

\[
\frac{dV_1^*}{dk_1} = \frac{d}{dk_1} \left( \frac{k_1}{\alpha^*} \right) = \frac{\alpha^* - k_1 \frac{d\alpha^*}{dk_1}}{[\alpha^*]^2}.
\]

(A.29)

Note that \(dV_1^*/dk_1 > 0\) when \(k_1 \to 0\). Thus, the equilibrium valuation \(V_1^*\) is decreasing in \(k_1\) when \(k_1\) is sufficiently small.

Next, suppose the entrepreneur’s effort \(e_1\) is exogenous, and define \(\rho_1 \equiv \rho_1(e_1)\). The early stage deal values are then given by

\[
D_1^E = \rho_1 [gU_2^E + (1 - g)(1 - \alpha)y_1] - c
\]

(A.30)

\[
D_1^A = \rho_1 [gU_2^A + (1 - g)\alpha y_1] - k_1
\]

(A.31)
where $c$ is the entrepreneur’s disutility of providing effort $e_1$. The optimal equity share for the angel, $\alpha^*$, then satisfies the symmetric Nash bargaining solution, which accounts for the outside option of each party ($U_1^E$ for the entrepreneur, and 0 for the angel because of free entry). Let $\tilde{D}_1^E$ and $\tilde{D}_1^A$ denote the deal values reflecting the Nash bargaining solution, which are given by

$$\tilde{D}_1^E = \frac{1}{2} \left[ \rho_1 \left( g (U_2^E + U_2^A) + (1 - g) y_1 \right) - k_1 - c + U_1^E \right]$$  
(A.32)

$$\tilde{D}_1^A = \frac{1}{2} \left[ \rho_1 \left( g (U_2^E + U_2^A) + (1 - g) y_1 \right) - k_1 - c - U_1^E \right].$$  
(A.33)

The equilibrium equity share for the angel, $\alpha^*$, then satisfies $D_1^E(\alpha^*) = \tilde{D}_1^E$ and $D_1^A(\alpha^*) = \tilde{D}_1^A$. Recall that $U_2^A = U_2^E$ in equilibrium. Thus,

$$\alpha^* = \frac{1}{2} + \frac{k_1 - c - U_1^E}{2 \rho_1 (1 - g) y_1}.$$  
(A.34)

The equilibrium early stage valuation is $V_1^* = k_1 / \alpha^*$. We get

$$\frac{dV_1^*}{dk_1} = \frac{\alpha^* - k_1}{[\alpha^*]^2}.$$  
(A.35)

The denominator is always non-negative. Moreover, note that $N \geq 0$ for $k_1 \to 0$, which implies that $dV_1^*/dk_1 \geq 0$ for $k_1 \to 0$. To show that $dV_1^*/dk_1 > 0$ for all $k_1 > 0$, it is thus sufficient to verify that $dN/dk_1 > 0$:

$$\frac{dN}{dk_1} = \frac{d\alpha^*}{dk_1} - \left( \frac{d\alpha^*}{dk_1} + k_1 \frac{d^2\alpha^*}{dk_1^2} \right) = -k_1 \frac{d^2\alpha^*}{dk_1^2}.$$  
(A.36)

We need to find the sign of $d^2\alpha^*/dk_1^2$. We start by taking the first derivative of $\alpha^*$ w.r.t. $k_1$:

$$\frac{d\alpha^*}{dk_1} = \frac{1}{2 \rho_1 (1 - g) y_1} \left[ 1 - \frac{dU_1^E}{dk_1} \right].$$  
(A.37)

It is easy to see that $\tilde{D}_1^E - U_1^E = \tilde{D}_1^A$. Thus, the condition defining $U_1^E$ simplifies to

$$U_1^E [r + \delta_1] - \frac{\phi_1^2}{\sigma_1^A} \left[ \tilde{D}_1^A \right]^2 + \sigma_1^E = 0.$$  
(A.38)

Thus,

$$\frac{dU_1^E}{dk_1} = -\frac{a_1 \tilde{D}_1^A}{r + \delta_1 + a_1 \tilde{D}_1^A},$$  
(A.39)

where $a_1 = \phi_1^2 / \sigma_1^A$. Consequently,

$$\frac{d\alpha^*}{dk_1} = \frac{1}{2 \rho_1 (1 - g) y_1} \left[ 1 + \frac{1}{(r + \delta_1) \left[ a_1 \tilde{D}_1^A \right]^{-1} + 1} \right].$$  
(A.40)
We then get
\[
\frac{d^2 \alpha^*}{dk_1^2} = \frac{1}{2 \rho_1 (1 - g) y_1} \left[ -\frac{1}{2} a_1 (r + \delta_1) \left[ a_1 \tilde{D}_1^A \right]^{-2} \left[ 1 + \frac{dU^E_1}{dk_1} \right] \right] \frac{\left[ (r + \delta_1) \left[ a_1 \tilde{D}_1^A \right]^{-1} + 1 \right]^2}{(r + \delta_1)}.
\] (A.41)

Note that
\[
1 + \frac{dU^E_1}{dk_1} = 1 - \frac{a_1 \tilde{D}_1^A}{r + \delta_1 + a_1 \tilde{D}_1^A} = \frac{r + \delta_1}{r + \delta_1 + a_1 \tilde{D}_1^A} > 0.
\] (A.42)

Thus, \(d^2 \alpha^*/dk_1^2 < 0\). This implies that \(dN/dk_1 > 0\), and therefore \(dV^*/dk_1 > 0\).

**Proof of Proposition 2.**

In equilibrium, \(U^E_2 = U^A_2\). Moreover, we will show in Proof of Proposition 3 that \(dU^E_2/d\phi_2 > 0\) and \(dU^E_2/d\delta_2, dU^E_2/d\sigma_2, dU^E_2/d\sigma_Y, dU^E_2/dk_2 < 0\). Consider the equilibrium degree of competition \(\theta^*_1\). With \(U^E_2 = U^A_2\) note that
\[
\frac{d\theta^*_1}{dU^E_2} = 2 \frac{\phi_1^2}{[\sigma_1^A]^2} D_1^A \frac{dD_1^A}{dU^E_2}.
\] (A.43)

For a given \(\alpha\) we find that
\[
\frac{dD_1^A}{dU^E_2} = \rho_1^1 (e_1^*) \frac{d\theta^*_1}{dU^E_2} \left[ gU^E_2 + (1 - g)\alpha y_1 \right] + \rho_1 (e_1^*) g > 0.
\] (A.44)

Moreover, applying the Envelope Theorem we get \(dD^E_1/dU^E_2 = g \rho_1 (e_1^*) > 0\). Thus, the bargaining frontier shifts outwards, so that \(dD^E_1/dU^E_2 > 0\) and \(dD^A_1/dU^E_2 > 0\) at the equilibrium equity share \(\alpha^*\). This implies that \(d\theta^*_1/dU^E_2 > 0\), and consequently, \(d\theta^*_1/d\phi_2 > 0\) and \(d\theta^*_1/d\delta_2, d\theta^*_1/d\sigma_2, d\theta^*_1/d\sigma_Y, d\theta^*_1/dk_2 < 0\).

Now consider the equilibrium inflow of entrepreneurs \(m^E_1 = F(U^E_1)\), with \(F'(U^E_1) > 0\).

Using Eq. (A.11) we get
\[
\frac{dU^E_1}{dU^E_2} = \frac{\phi_1^1 (e_1^*)}{\sigma_1^A} \frac{d\theta^*_1}{dU^E_2} \frac{dU^E_2}{U^E_2} \frac{d\alpha^*}{dU^E_2} + \frac{\partial \Gamma}{\partial U^E_2},
\] (A.45)

where \(\Gamma = D_1^A \left[ D^E_1 - U^E_1 \right]\). Note that
\[
\frac{d\Gamma}{dU^E_2} = \frac{d\Gamma}{d\theta^*_1} \frac{d\theta^*_1}{dU^E_2} + \frac{d\Gamma}{d\alpha^*} \frac{d\alpha^*}{dU^E_2} + \frac{\partial \Gamma}{\partial U^E_2},
\] (A.46)
where \( de^*_i/dU^E_2 > 0 \) and \( d\Gamma/d\alpha = 0 \) (see Eq. (A.1)). Moreover,

\[
\frac{d\Gamma}{de_1} = \frac{dD^A_1}{de_1} \left[ D^E_1 - U^E_1 \right] + D^A_1 \frac{D^E_1}{de_1} = \rho'_1(e^*_1) \left[gU^A_2 + (1 - g)\alpha y_1\right] \left[D^E_1 - U^E_1\right] > 0
\]

\( > 0 \)

\[
\frac{\partial\Gamma}{\partial U^E_2} = \frac{\partial D^A_1}{\partial U^E_2} \left[D^E_1 - U^E_1\right] + D^A_1 \frac{\partial D^E_1}{\partial U^E_2} > 0
\]

(A.48)

This implies that \( du^E_1 / du^E_2 > 0 \), and therefore, \( dF(U^E_1)/du^E_2 > 0 \). Our comparative statics results for \( U^E_2 \) (see Proof of Proposition 3) then imply that \( dm^E_1/d\phi_2 > 0 \) and \( dm^E_1/d\delta_2 \), \( dm^E_1/d\sigma_2, dm^E_1/d\sigma_Y, dm^E_1/dk_2 < 0 \).

Next consider the equilibrium inflow of angels, \( m^A^*_1 \), which is defined by

\[
m^A^*_1 = x^*_1 = F(U^E_1) \frac{\phi_1\sqrt{\theta_i^1}}{\delta_1 + \phi_1\sqrt{\theta_i^1}}.
\]

(A.49)

One can show that \( dT/d\sqrt{\theta^1_i} > 0 \). Our comparative statics results for \( m^E_1 \) and \( \theta^*_1 \) then imply that \( dm^A^*_1/d\phi_2 > 0 \) and \( dm^A^*_1/d\delta_2, dm^A^*_1/d\sigma_2, dm^A^*_1/d\sigma_Y, dm^A^*_1/dk_2 < 0 \).

Now consider the equilibrium equity share \( \alpha^* \) for angels. Recall that \( dD^E_1/dU^E_2 > 0 \) and \( dD^A_1/dU^E_2 > 0 \) at the equilibrium equity share \( \alpha^* \). Moreover, using the Envelope Theorem it is straightforward to show that \( dD^A_1/dU^E_2 > dD^E_1/dU^E_2 \). The Nash bargaining solution then implies that \( d\alpha^*/du^E_2 < 0 \). Thus, \( d\alpha^*/d\phi_2 < 0 \) and \( d\alpha^*/d\delta_2, d\alpha^*/d\sigma_2, d\alpha^*/d\sigma_Y, d\alpha^*/dk_2 > 0 \). For the equilibrium valuation \( V^*_1 = k_1/\alpha^* \) we can then infer that \( dV^*_1/d\phi_2 > 0 \) and \( dV^*_1/d\delta_2, dV^*_1/d\sigma_2, dV^*_1/d\sigma_Y, dV^*_1/dk_2 < 0 \).

Finally consider the equilibrium success rate \( \rho^*_1(e^*_1) \), with \( \rho'_1(e^*_1) > 0 \). Using Eq. (3) it is straightforward to show that \( \partial e^*_1/\partial U^E_2 > 0 \) and \( \partial e^*_1/\partial \alpha < 0 \). Using our comparative statics results for \( U^E_2 \) and \( \alpha^* \) we can then infer that \( de^*_1/d\phi_2 > 0 \) and \( de^*_1/d\delta_2, de^*_1/d\sigma_2, de^*_1/d\sigma_Y, de^*_1/dk_2 < 0 \). Consequently, \( dp^*_1(e^*_1)/d\phi_2 > 0 \) and \( dp^*_1(e^*_1)/d\delta_2, dp^*_1(e^*_1)/d\sigma_2, dp^*_1(e^*_1)/d\sigma_Y, dp^*_1(e^*_1)/dk_2 < 0 \).
VC market: derivation of deal values and equity shares.

Let $CV_i$ denote the value generated by the coalition $i = EAV, EV, EA, AV, E, A, V$. Using the Shapley value we get the following general deal values from the tripartite bargaining game:

$$D_E^2 = \frac{1}{3} [CV_{EAV} - CV_{AV}] + \frac{1}{6} [CV_{EA} - CV_A] + \frac{1}{6} [CV_{EV} - CV_V] + \frac{1}{3} CV_E \quad (A.50)$$

$$D_A^2 = \frac{1}{3} [CV_{EAV} - CV_{EV}] + \frac{1}{6} [CV_{EA} - CV_E] + \frac{1}{6} [CV_{AV} - CV_A] + \frac{1}{3} CV_A \quad (A.51)$$

$$D_V^2 = \frac{1}{3} [CV_{EAV} - CV_{EA}] + \frac{1}{6} [CV_{EV} - CV_E] + \frac{1}{6} [CV_{AV} - CV_A] + \frac{1}{3} CV_V \quad (A.52)$$

We note that $CV_{EAV} = \pi$ and $CV_{AV} = CV_{EV} = CV_E = CV_A = CV_V = 0$. Moreover, by assumption we have $U_{E^2} + U_{A^2} > y_1$, so that $CV_{EA} = U_{E^2} + U_{A^2}$. Thus,

$$D_E^2 = \frac{1}{3} \pi + \frac{1}{6} [U_{E^2} + U_{A^2}] \quad (A.53)$$

$$D_A^2 = \frac{1}{3} \pi + \frac{1}{6} [U_{E^2} + U_{A^2}] \quad (A.54)$$

$$D_V^2 = \frac{1}{3} \pi - \frac{1}{3} [U_{E^2} + U_{A^2}] \quad (A.55)$$

The deal values then allow us to derive the equilibrium equity shares $\beta^{E*}$, $\beta^{A*}$, and $\beta^{V*}$. The equilibrium equity share for entrepreneurs, $\beta^{E*}$, ensures that their actual net payoff equals their deal value from the bargaining game: $\beta^{E*} y_2 = D_E^2$. Solving this for $\beta^{E*}$ yields

$$\beta^{E*} = \frac{D_E^2}{y_2} = \frac{1}{6 y_2} \left[2 \pi + U_{E^2} + U_{A^2} \right] \cdot (A.56)$$

Likewise we get

$$\beta^{A*} = \frac{D_A^2}{y_2} = \frac{1}{6 y_2} \left[2 \pi + U_{E^2} + U_{A^2} \right] \quad (A.57)$$

$$\beta^{V*} = \frac{k_2 + D_V^2}{y_2} = \frac{1}{3 y_2} \left[3 k_2 + \pi - (U_{E^2} + U_{A^2}) \right] \cdot (A.58)$$

Derivation of VC market equilibrium.

The first part of the derivation follows along the lines of the derivation of the angel market equilibrium: Using Eq. (13) we get $\theta_2^* = \left[\phi_2 D_V^2 / \sigma_2^2 \right]^2$. Moreover, using Eq. (14) and the relationship $M_2^{V*} = M_2^{E*} \theta_2^*$ we find

$$M_2^{V*} = g \rho_1 (e_1^*) x_1^* \frac{\theta_2^*}{\delta_2 + \phi_2 \sqrt{\theta_2^*}} \cdot (A.59)$$
Using $M_2^{E*} = M_2^{V*}/\theta_2^*$ and the definition of $M_2^{V*}$, we can write $x_2^*$ as

$$x_2^* = \phi_2 \left[ M_2^{V*} M_2^{E*} \right]^{0.5} = \frac{\phi_2 M_2^{V*}}{\sqrt{\theta_2^*}} = m_2^{E*} \frac{\phi_2 \sqrt{\theta_2^*}}{\delta_2 + \phi_2 \sqrt{\theta_2^*}},$$

(A.60)

where $m_2^{E*} = g \rho_1(e_1^*) x_1^*$. Furthermore, using Eq. (15) and $q_2^V = x_2/M_2^V$ we find that $m_2^{V*} = q_2^V M_2^{V*} = x_2^*$.

Finally, using the equilibrium equity share $\beta^V*$ for VCs we can write $V_2^*$ as follows:

$$V_2^* = \frac{k_2}{\beta^V*} = \frac{k_2 y_2}{k_2 + D_2^V} = \left( \frac{3k_2}{3k_2 + \pi - (U_2^E + U_2^A)} \right) y_2.$$

(A.61)

**Proof of Proposition 3.**

First we need to derive a condition which defines $U_2^E$. We can write Eq. (12) as

$$U_2^E \left[ r + \delta_2 \right] = -\sigma_2 + q_2^E \left[ D_2^E - U_2^E \right].$$

(A.62)

Note that $D_2^E - U_2^E = \pi/3 - 2U_2^E/3 = D_2^V$. Using $q_2^E = \phi_2 \left[ M_2^{V*}/M_2^{E*} \right]^{0.5} = \phi_2 \sqrt{\theta_2^*} = \phi_2^3 D_2^V / \sigma_2^2$, we get the following condition which defines $U_2^E$:

$$U_2^E \left[ r + \delta_2 \right] - \frac{\phi_2^2}{\sigma_2^2} \left[ D_2^V \right]^2 + \sigma_2 = 0.$$

(A.63)

Consider the equilibrium degree of competition $\theta_2^*$. Recall that $U_2^A = U_2^E$ in equilibrium; thus,

$$\frac{d\theta_2^*}{dU_2^A} = \frac{d\theta_2^*}{dU_2^E} = \frac{2 \phi_2^3 D_2^V}{\sigma_2^2} \frac{dD_2^V}{dU_2^E} = -\frac{4 \phi_2^3 D_2^V}{3 \sigma_2^2} < 0.$$

(A.64)

Note that $\delta_2$ only affects $U_2^E$ in the definition of $\theta_2^*$. Implicitly differentiating $U_2^E$ w.r.t. $\delta_2$ yields

$$\frac{dU_2^E}{d\delta_2} = \frac{U_2^E}{r + \delta_2 + 4 \phi_2^3 D_2^V} < 0,$$

(A.65)

which implies that $d\theta_2^*/d\delta_2 > 0$. Likewise, $\sigma_2$ only affects $U_2^E$ in the definition of $\theta_2^*$. We get

$$\frac{dU_2^E}{d\sigma_2} = -\frac{1}{r + \delta_2 + 4 \phi_2^3 D_2^V} < 0.$$

(A.66)

Thus, $d\theta_2^*/d\sigma_2 > 0$. Next, differentiating $U_2^E$ w.r.t. $\phi_2$ yields

$$\frac{d\theta_2^*}{d\phi_2} = 2 \phi_2 D_2^V \left[ \frac{D_2^V}{\sigma_2^2} + \phi_2 \frac{dD_2^V}{dU_2^E} \frac{dU_2^E}{d\phi_2} \right] = 2 \phi_2 D_2^V \left[ \frac{D_2^V}{\sigma_2^2} - \frac{2}{3} \phi_2 \frac{dU_2^E}{d\phi_2} \right],$$

(A.67)

with

$$\frac{dU_2^E}{d\phi_2} = \frac{2 \phi_2^3 \left[ D_2^V \right]^2}{r + \delta_2 + 4 \phi_2^3 D_2^V} > 0.$$

(A.68)
Therefore,
\[
\frac{d\theta^*_2}{d\phi_2} = \frac{2\phi_2 D_2^V}{\sigma_2^2} \frac{(r + \delta_2)D_2^V}{r + \delta_2 + \frac{4\phi_2^3}{3\sigma_2^2} D_2^V} > 0.
\] (A.69)

Likewise,
\[
\frac{d\theta^*_2}{d\sigma_2^V} = \frac{2\phi_2^2 D_2^V}{\sigma_2^V} \left[ -\frac{2}{3} \frac{dU_2^E}{d\sigma_2^V} \sigma_2^V - D_2^V \right], \quad \text{with} \quad \frac{dU_2^E}{d\sigma_2^V} = -\frac{\phi_2^3 D_2^V}{\sigma_2^2} \frac{1}{r + \delta_2 + \frac{4\phi_2^3}{3\sigma_2^2} D_2^V} < 0.
\] (A.70)

Consequently,
\[
\frac{d\theta^*_2}{d\sigma_2^V} = -\frac{2\phi_2^2 D_2^V}{\sigma_2^V} \frac{1}{\sigma_2^2} \left[ \frac{2}{3\sigma_2^4} \left( D_2^V \right)^2 + (r + \delta_2)D_2^V \right] < 0.
\] (A.71)

Moreover, we get
\[
\frac{d\theta^*_2}{dk_2} = \frac{2\phi_2^2 D_2^Y}{\sigma_2^2} \left[ \frac{1}{3} - \frac{2}{3} \frac{dU_2^E}{dk_2} \right], \quad \text{with} \quad \frac{dU_2^E}{dk_2} = -\frac{\phi_2^3 D_2^Y}{\sigma_2^2} \frac{1}{r + \delta_2 + \frac{4\phi_2^3}{3\sigma_2^2} D_2^Y} < 0.
\] (A.72)

Thus,
\[
\frac{d\theta^*_2}{dk_2} = \frac{2\phi_2^2 D_2^Y}{3\sigma_2^2} \frac{r + \delta_2}{r + \delta_2 + \frac{4\phi_2^3}{3\sigma_2^2} D_2^Y} < 0.
\] (A.73)

Next, recall that \(m_2^V = x_2^*\) is given by
\[
m_2^V = x_2^* = \frac{g\rho_1(e_1^*)x_2^*}{\delta_2 + \phi_2\sqrt{\theta_2^*}} \equiv m_2^{x_2}\) (A.74)

We have shown in Proof of Proposition 2 that \(dx_2^*/d\phi_2 > 0\) and \(dx_2^*/d\delta_2, dx_2^*/d\sigma_2, dx_2^*/d\sigma_2^V, dx_2^*/dk_2 < 0\). Likewise, we have shown that \(d\rho_1(e_1^*)/d\phi_2 > 0\) and \(d\rho_1(e_1^*)/d\delta_2, d\rho_1(e_1^*)/d\sigma_2, d\rho_1(e_1^*)/d\sigma_2^V, d\rho_1(e_1^*)/dk_2 < 0\). Moreover, it is straightforward to verify that \(dT/d(\phi_2 \sqrt{\theta_2^*}) > 0\). Using our comparative statics results for \(\theta_2^*\), we can infer that \(dT/d\phi_2, dT/d\delta_2, dT/d\sigma_2 > 0\), and \(dT/d\sigma_2^V, dT/dk_2 < 0\). All this implies that \(dm_2^V/d\phi_2 > 0\) and \(dm_2^V/d\sigma_2^V, dm_2^V/dk_2 < 0\), while the effects of \(\delta_2\) and \(\sigma_2\) on \(m_2^V\) are ambiguous.

Now consider the equilibrium late stage valuation \(V_2^*\):
\[
V_2^* = \left( \frac{3k_2}{3k_2 + \pi - 2U_2^E} \right) y_2.
\] (A.75)
Recall that \( dU_2^E/d\phi_2 > 0 \), and \( dU_2^E/d\sigma_2 \), \( dU_2^E/d\sigma_2^A \), \( dU_2^E/d\delta_2 < 0 \). Thus, \( dV_2^*/d\phi_2 > 0 \) and \( dV_2^*/d\sigma_2 \), \( dV_2^*/d\sigma_2^A \), \( dV_2^*/d\delta_2 < 0 \). Furthermore, recall that \( V_2^* \) can also be written as \( V_2^* = k_2y_2/(k_2 + D_2^V) \). Taking the first derivative of \( V_2^* \) w.r.t. \( k_2 \) yields

\[
\frac{dV_2^*}{dk_2} = \frac{k_2 + D_2^V - k_2 \left[ 1 - \frac{1}{3} - \frac{2}{3} \frac{dU_2^E}{dk_2} \right] y_2}{[k_2 + D_2^V]^2} = \frac{\frac{1}{3} k_2 + D_2^V + \frac{2}{3} \frac{dU_2^E}{dk_2}}{[k_2 + D_2^V]^2} y_2.
\]  

(A.76)

Note that the denominator is always positive. Moreover, we have \( dN/dk_2 \) and \( N \rightarrow 0 \). For all \( k_2 > 0 \), it is sufficient to show that \( dN/dk_2 > 0 \):

\[
\frac{dN}{dk_2} = \frac{1}{3} - \frac{1}{3} - \frac{2}{3} \frac{dU_2^E}{dk_2} + \frac{2}{3} \left[ \frac{dU_2^E}{dk_2} + k_2 \frac{d^2U_2^E}{dk_2^2} \right] = \frac{2}{3} k_2 \frac{d^2U_2^E}{dk_2^2}.
\]  

(A.77)

It remains to identify the sign of \( d^2U_2^E/dk_2^2 \). Using \( a_2 \equiv \phi_2^\delta/\sigma_2^V \) we can write \( dU_2^E/dk_2 \) as

\[
\frac{dU_2^E}{dk_2} = -\frac{2}{3} a_2 D_2^V \frac{(r + \delta_2 + \frac{4}{3} a_2 D_2^V)}{(r + \delta_2) [a_2 D_2^V]^{-1} + \frac{4}{3}}.
\]  

(A.78)

Thus,

\[
\frac{d^2U_2^E}{dk_2^2} = \frac{2}{3} a_2 (r + \delta_2) [a_2 D_2^V]^{-2} \left[ 1 + 2 \frac{dU_2^E}{dk_2} \right].
\]  

(A.79)

Note that

\[
1 + 2 \frac{dU_2^E}{dk_2} = 1 - \frac{4}{3} a_2 D_2^V \frac{r + \delta_2 + \frac{4}{3} a_2 D_2^V}{r + \delta_2 + \frac{4}{3} a_2 D_2^V} > 0.
\]  

(A.80)

Hence, \( d^2U_2^E/dk_2^2 > 0 \), so that \( dN/dk_2 > 0 \). Consequently, \( dV_2^*/dk_2 > 0 \).

\[ \square \]

**Proof of Proposition 4.**

We can see from Eq. (A.63) that \( U_2^E \) (and therefore \( U_2^A \)) does not depend on the early stage parameters \( \phi_1, \delta_1, \sigma_1^E, \sigma_1^A \), and \( k_1 \). This also implies that \( D_2^V \), and therefore \( \theta_2^V \) and \( V_2^* \), do not depend on these parameters.

Now consider the equilibrium inflow of start-ups \( m_2^{x_*} = g\rho_1(e_1^*)x_1^* \). Recall from Proposition 1 that \( dx_1^*/d\phi_1 > 0 \) and \( dx_1^*/d\sigma_1^A \), \( dx_1^*/dk_1 < 0 \), while the effects of \( \delta_1 \) and \( \sigma_1^E \) are ambiguous.

Moreover, we know from Proposition 1 that \( d\rho_1(e_1^*)/d\phi_1 > 0 \) and \( d\rho_1(e_1^*)/d\delta_1, d\rho_1(e_1^*)/d\sigma_1^E, d\rho_1(e_1^*)/d\sigma_1^A, d\rho_1(e_1^*)/dk_1 < 0 \). This implies that \( dm_2^{x_*}/d\phi_1 > 0 \) and \( dm_2^{x_*}/d\sigma_1^A, dm_2^{x_*}/dk_1 < 0 \), while the effects of \( \delta_1 \) and \( \sigma_1^E \) are ambiguous.

Finally consider the equilibrium inflow of VCs \( m_2^{V_*} \), as defined by

\[
m_2^{V_*} = x_2^* = m_2^{x_*} \frac{\phi_2\sqrt{\theta_2^*}}{\delta_2 + \phi_2\sqrt{\theta_2^*}}.
\]  

(A.81)
Recall that $\theta_2^*$ does not depend on the early stage parameters. Our comparative statics results for $m_2^{E*}$ then imply that $dm_2^{V*}/d\phi_1 > 0$ and $dm_2^{V*}/d\sigma_1^2$, $dm_2^{V*}/dk_1 < 0$, while the effects of $\delta_1$ and $\sigma_E^2$ are ambiguous.

\[ \Box \]

**Angel protection: derivation of deal values and equity shares.**

The new coalition values are given by $CV_{EAV} = \pi$, $CV_{EA} = U_2^E + U_2^A$, $CV_{EV} = \lambda \pi$, and $CV_{AV} = CV_E = CV_A = CV_V = 0$. Using the general deal values (A.50), (A.51), and (A.52), we get

\[
D_2^E = \frac{1}{6} [2 + \lambda] \pi + \frac{1}{6} [U_2^E + U_2^A] \quad (A.82)
\]

\[
D_2^A = \frac{1}{3} [1 - \lambda] \pi + \frac{1}{6} [U_2^E + U_2^A] \quad (A.83)
\]

\[
D_2^V = \frac{1}{6} [2 + \lambda] \pi - \frac{1}{3} [U_2^E + U_2^A] \quad (A.84)
\]

The new equilibrium equity share for entrepreneurs, $\beta_{E*}$, ensures that their actual net payoff equals their deal value from the bargaining game: $\beta_{E*} y_2 = D_2^E$. Solving this for $\beta_{E*}$ yields

\[
\beta_{E*} = \frac{D_2^E}{y_2} = \frac{1}{6y_2} \left[ (2 + \lambda) \pi + U_2^E + U_2^A \right]. \quad (A.85)
\]

Likewise we get

\[
\beta_{A*} = \frac{D_2^A}{y_2} = \frac{1}{6y_2} \left[ 2 (1 - \lambda) \pi + U_2^E + U_2^A \right] \quad (A.86)
\]

\[
\beta_{V*} = \frac{k_2 + D_2^V}{y_2} = \frac{1}{6y_2} \left[ 6k_2 + (2 + \lambda) \pi - 2 (U_2^E + U_2^A) \right]. \quad (A.87)
\]

**Proof of Proposition 5.**

We first show that $dU_2^A/d\lambda < 0$. Note that $D_2^A \neq D_2^E$ for $\lambda > 0$, and recall that $q_2^E = \phi_2 \left( M_2^{V*}/M_2^{E*} \right)^{0.5} = \phi_2^2 D_2^V/\sigma_2^V$. Thus, using Eq. (12) we define

\[
F \equiv U_2^E (r + \delta_2) + \sigma - a_2 D_2^V \left[ D_2^E - U_2^E \right] = 0 \quad (A.88)
\]

\[
G \equiv U_2^A (r + \delta_2) + \sigma - a_2 D_2^V \left[ D_2^A - U_2^A \right] = 0, \quad (A.89)
\]

where $a_2 = \phi_2^2/\sigma_2^V$. Using Cramer’s rule we get

\[
\frac{dU_2^A}{d\lambda} = \frac{-\frac{\partial F}{\partial x} \frac{\partial G}{\partial u_2^E} - \frac{\partial G}{\partial x} \frac{\partial F}{\partial u_2^A}}{\frac{\partial F}{\partial u_2^A} \frac{\partial G}{\partial u_2^E} - \frac{\partial G}{\partial u_2^A} \frac{\partial F}{\partial u_2^E}} = \frac{-\frac{\partial F}{\partial x} \frac{\partial G}{\partial u_2^E} + \frac{\partial G}{\partial x} \frac{\partial F}{\partial u_2^A}}{\frac{\partial F}{\partial u_2^A} \frac{\partial G}{\partial u_2^E} - \frac{\partial G}{\partial u_2^A} \frac{\partial F}{\partial u_2^E}}. \quad (A.90)
\]
The denominator is negative if
\[
\frac{\partial G}{\partial U_2^2} \frac{\partial F}{\partial U_2^2} > \frac{\partial F}{\partial U_2^2} \frac{\partial G}{\partial U_2^2}, \tag{A.91}
\]
which is equivalent to
\[
\left[ r + \delta_2 + \frac{1}{6} a_2 \left[ 2 \left( D_2^A - U_2^A \right) + 5 D_2^Y \right] \right] \left[ r + \delta_2 + \frac{1}{6} a_2 \left[ 2 \left( D_2^E - U_2^E \right) + 5 D_2^Y \right] \right] > \frac{1}{6} a_2 \left[ 2 \left( D_2^E - U_2^E \right) - D_2^Y \right] \frac{1}{6} a_2 \left[ 2 \left( D_2^A - U_2^A \right) - D_2^Y \right]. \tag{A.92}
\]
If this condition holds for \( r + \delta_2 = 0 \), then it also holds for all \( r + \delta_2 > 0 \). Setting \( r + \delta_2 = 0 \) we get
\[
10 \left( D_2^A - U_2^A \right) D_2^Y + 10 D_2^Y \left( D_2^E - U_2^E \right) + 24 \left( D_2^V \right)^2 > -2 \left( D_2^E - U_2^E \right) D_2^Y - 2 \left( D_2^A - U_2^A \right) D_2^Y. \tag{A.93}
\]
This condition is satisfied as \( D_2^E > U_2^E \) and \( D_2^A > U_2^A \). Thus, the denominator of \( dU_2^A/d\lambda \) is strictly negative. Likewise, the numerator is positive if
\[
\frac{\partial G}{\partial \lambda} \frac{\partial F}{\partial U_2^2} > \frac{\partial F}{\partial \lambda} \frac{\partial G}{\partial U_2^2}, \tag{A.94}
\]
which is equivalent to
\[
\frac{1}{6} \pi a_2 \left[ \left( D_2^A - U_2^A \right) - 2 D_2^Y \right] \left[ r + \delta_2 + \frac{1}{6} a_2 \left[ 2 \left( D_2^E - U_2^E \right) + 5 D_2^Y \right] \right] < \frac{1}{6} a_2 \left[ \left( D_2^E - U_2^E \right) + D_2^Y \right] \frac{1}{6} a_2 \left[ 2 \left( D_2^A - U_2^A \right) - D_2^Y \right]. \tag{A.95}
\]
This condition can be written as
\[
\frac{2}{a_2} \left( r + \delta_2 \right) \left[ \left( D_2^A - U_2^A \right) - 2 D_2^Y \right] + \frac{D_2^A - U_2^A}{a_2} D_2^Y - D_2^Y \left[ D_2^E - U_2^E \right] < 3 \left[ D_2^Y \right]^2. \tag{A.96}
\]
From \( F \) and \( G \) we know that
\[
D_2^Y \left[ D_2^E - U_2^E \right] = \frac{U_2^E \left( r + \delta_2 \right) + \sigma}{a_2} \quad \text{and} \quad D_2^V \left[ D_2^A - U_2^A \right] = \frac{U_2^A \left( r + \delta_2 \right) + \sigma}{a_2}, \tag{A.97}
\]
so that we can write condition (A.96) as follows:
\[
\frac{2}{a_2} \left( r + \delta_2 \right) \left[ \left( D_2^A - U_2^A \right) - 2 D_2^Y \right] + \frac{U_2^A \left( r + \delta_2 \right) + \sigma}{a_2} - \frac{U_2^E \left( r + \delta_2 \right) + \sigma}{a_2} < 3 \left[ D_2^Y \right]^2 \tag{A.98}
\]
\[
\Leftrightarrow \quad \left( r + \delta_2 \right) \left[ 2 D_2^A - U_2^A - 4 D_2^Y - U_2^E \right] < 3 \left[ D_2^V \right]^2 a_2. \tag{A.99}
\]
We now show that $T < 0$. Using the definitions of $D^A_2$ and $D^V_2$ we can write $T < 0$ as

$$
\frac{2}{3} [1 - \lambda] \pi + \frac{1}{3} [U^E_2 + U^A_2] - U^A_2 - \frac{2}{3} [2 + \lambda] \pi + \frac{4}{3} [U^E_2 + U^A_2] - U^E_2 < 0 \quad (A.100)
$$

$$
\Leftrightarrow \quad U^E_2 + U^A_2 < [1 + 2\lambda] \pi. \quad (A.101)
$$

This condition is satisfied for all $\lambda \geq 0$ because $\pi > U^E_2 + U^A_2$. Thus, the numerator of $dU^A_2 / d\lambda$ is strictly positive. Consequently, $dU^A_2 / d\lambda < 0$. Finally note that $\partial D^E_2 / \partial \lambda = \pi / 6 < |\partial D^A_2 / \partial \lambda| = \pi / 3$. Thus, $d[U^E_2 + U^A_2] / d\lambda < 0$, which implies that $dD^V_2 / d\lambda > 0$.

Next we analyze the effects of $\lambda$ on the early stage equilibrium variables. Consider the equilibrium degree of competition $\theta^*_1$. We get

$$
\frac{d\theta^*_1}{d\lambda} = 2 \frac{\phi^2}{[\sigma^A_1]^2} D^A_1 \frac{dD^A_1}{d\lambda}. \quad (A.102)
$$

Recall that

$$
\frac{d}{d\lambda} (U^A_2 + U^E_2) = \frac{dU^A_2}{d\lambda} + \frac{dU^E_2}{d\lambda} < 0. \quad (A.103)
$$

This implies

$$
\frac{dD^A_1}{d\lambda} + \frac{dD^E_1}{d\lambda} < 0 \quad \Rightarrow \quad \frac{dD^A_1}{d\lambda} < 0. \quad (A.104)
$$

Thus, $d\theta^*_1 / d\lambda < 0$.

Now consider the equilibrium entry of entrepreneurs $m^E_1$. Using Eq. (A.11), we get

$$
\frac{dU^E_1}{d\lambda} = \frac{\phi^2}{\sigma^A_1} \left[ \frac{dD^A_1}{d\lambda} \left[ D^E_1 - U^E_1 \right] + D^A_1 \frac{dD^E_1}{d\lambda} \right] \frac{1}{r + \delta_1 - \frac{\phi^2}{\sigma^A_1} \left[ \frac{d\Gamma}{d\alpha} \frac{d\alpha^*}{d\alpha} + \frac{\partial \Gamma}{\partial U^E_1} \right]}, \quad (A.105)
$$

where $\Gamma = D^A_1 \left[ D^E_1 - U^E_1 \right]$. Note that $d\Gamma / d\alpha = 0$; see Eq. (A.1). Thus,

$$
\frac{dU^E_1}{d\lambda} = \frac{\phi^2}{\sigma^A_1} \left[ \frac{dD^A_1}{d\lambda} \left[ D^E_1 - U^E_1 \right] + D^A_1 \frac{dD^E_1}{d\lambda} \right] \frac{1}{r + \delta_1 + \frac{\phi^2}{\sigma^A_1} D^A_1}, \quad (A.106)
$$

where the denominator is positive. Consequently, $dU^E_1 / d\lambda < 0$ if

$$
\frac{dD^A_1}{d\lambda} \left[ D^E_1 - U^E_1 \right] + D^A_1 \frac{dD^E_1}{d\lambda} < 0. \quad (A.107)
$$

Using Eq. (A.1) we can derive the following expression for $D^E_1 - U^E_1$:

$$
D^E_1 - U^E_1 = -\frac{\frac{dD^E_1}{d\alpha}}{\frac{dD^A_1}{d\alpha}} D^A_1, \quad (A.108)
$$
so that Eq. (A.107) can be written as

$$\frac{dD}{d\lambda} \left( -\frac{dD^E}{d\alpha} \right) + \frac{dD^F}{d\lambda} \equiv X < 0. \quad (A.109)$$

Recall that $d(D^A + D^E)/d\lambda < 0$, with $dD^A/d\lambda < 0$; thus, this condition is satisfied when $X \geq 1$. Note that $dD^E/d\alpha < 0$ and $dD^A/d\alpha > 0$. Hence, $X \geq 1$ if

$$0 \geq \frac{dD^A}{d\alpha} + \frac{dD^E}{d\alpha} = \frac{d}{d\alpha} [D^A + D^E]. \quad (A.110)$$

It is easy to show that the joint surplus is maximized when $\alpha = 0$ (which maximizes effort incentives for the entrepreneur); thus

$$\frac{d [D^A + D^E]}{d\alpha} \bigg|_{\alpha = \alpha^* > 0} < 0, \quad (A.111)$$

so that $X \geq 1$. Consequently, $dU^E_1/d\lambda < 0$, and therefore $dm^E_1/d\lambda = dF(U^E_1)/d\lambda < 0$.

Next consider the equilibrium inflow of angels, $m^A_1$, which is defined by

$$m^A_1 = x^*_1 = F(U^E_1) \underbrace{\frac{\phi_1 \sqrt{\theta^*_1}}{\delta_1 + \phi_1 \sqrt{\theta^*_1}}}_{= T}. \quad (A.112)$$

Note that $dT/d\sqrt{\theta^*_1} > 0$. Our comparative statics results for $m^E_1$ and $\theta^*_1$ then imply that $dm^A_1/d\lambda = dx^*_1/d\lambda < 0$.

Now consider the angel’s equilibrium equity share $\alpha^*$, which is defined by Eq. (A.1). We get

$$\frac{d\alpha^*}{d\lambda} = \frac{\alpha^*}{dU^E_2} \frac{dU^E_2}{d\lambda} + \frac{\alpha^*}{dU^A_2} \frac{dU^A_2}{d\lambda}, \quad (A.113)$$

where $dU^E_2/d\lambda > 0$ and $dU^A_2/d\lambda < 0$. Moreover, the Nash bargaining solution implies that $d\alpha^*/dU^E_2 > 0$ and $d\alpha^*/dU^A_2 < 0$. Thus, $d\alpha^*/d\lambda > 0$. For the equilibrium valuation $V^*_1 = k_1/\alpha^*$ this concurrently implies that $dV^*_1/d\lambda < 0$. Finally we know that $dD^E_1/d\lambda > 0$ in equilibrium. Using the Envelope Theorem we get

$$\frac{dD^E}{d\lambda} = \rho_1(e_1) \frac{d}{d\lambda} \left[ gU^E_2 + (1 - g)(1 - \alpha^*)y_1 \right] > 0, \quad (A.114)$$

which implies that $T > 0$. Using Eq. (3) we find

$$\frac{de^*_1}{d\lambda} = -\rho'_1(e_1) \frac{d}{d\lambda} \left[ gU^E_2 + (1 - g)(1 - \alpha)y_1 \right] + \mathcal{T}, \quad (A.115)$$
where \( T > 0 \), and the denominator is negative due to the second-order condition for \( e_1^* \). Thus, \( de_1^*/d\lambda > 0 \). This in turn implies that \( dp_1(e_1^*)/d\lambda > 0 \).

Finally we analyze the effects of \( \lambda \) on the late stage equilibrium variables. Note that \( d \left( U_2^E + U_2^A \right) /d\lambda < 0 \) also implies that \( d\hat{D}_2^Y /d\lambda > 0 \). Using the definitions of \( \theta_2^*, \beta V^* \) and \( V_2^* \), we can then infer that \( d\theta_2^*/d\lambda > 0 \), \( d\beta V^*/d\lambda > 0 \) and \( dV_2^*/d\lambda < 0 \). Moreover,

\[
\frac{dm_2^{E*}}{d\lambda} = \frac{d}{d\lambda} \left[ g\rho_1(e_1^*)x_1^* \right] = g \left[ \rho_1'(e_1^*) \frac{de_1^*}{d\lambda} x_1^* + \rho_1(e_1^*) \frac{dx_1^*}{d\lambda} \right]. \tag{A.116}
\]

In general, the effect on \( m_2^{E*} \) is ambiguous as \( de_1^*/d\lambda > 0 \) and \( dx_1^*/d\lambda < 0 \). However, we can see that \( dm_2^{E*}/d\lambda < 0 \) when \( \rho_1'(e_1^*) \to 0 \). Moreover, for \( \delta_1 \to 0 \) we have \( m_1^{A*} = m_2^{E*} \); with \( m_2^{E*} \) being sufficiently inelastic, we have \( dx_1^*/d\lambda \to 0 \), so that \( dm_2^{E*}/d\lambda > 0 \). Next, recall that \( m_2^{V*} \) is defined by

\[
m_2^{V*} = x_2^* = m_2^{E*} \frac{\phi_2 \sqrt{\theta_2^*}}{\delta_2 + \phi_2 \sqrt{\theta_2^*}}. \tag{A.117}
\]

One can show that \( dT/d\sqrt{\theta_2^*} > 0 \), so that \( dT/d\lambda > 0 \). Recall, however, that the sign of \( dm_2^{E*}/d\lambda \) is ambiguous. Thus, the effect of \( \lambda \) on \( m_2^{V*} = x_2^* \) is also ambiguous.

\[\Box\]

**Angel protection – angel not required for VC search.**

Suppose the entrepreneur can search for a VC without the angel. The entrepreneur then incurs the search cost \( \gamma \sigma \), with \( \gamma > 2 \). Using Nash bargaining, the deal values for the VC \( \hat{D}_2^Y \) and the entrepreneur \( \hat{D}_2^E \) are then given by

\[
\hat{D}_2^Y = \frac{1}{2} \left[ \lambda \pi - \hat{U}_2^E \right] \quad \hat{D}_2^E = \frac{1}{2} \left[ \lambda \pi + \hat{U}_2^E \right], \tag{A.118}
\]

where \( \hat{U}_2^E \) denotes the entrepreneur’s outside option.

Now consider the bargaining problem at the late stage between entrepreneur, angel, and VC. The new coalition values are given by \( CV_{EAV} = \pi \), \( CV_{EA} = U_2^E + U_2^A \), \( CV_{EV} = \lambda \pi \), \( CV_E = \hat{U}_2^E \), and \( CV_{AV} = CV_A = CV_V = 0 \). Using the Shapley value we then get the following deal values:

\[
D_2^E = \frac{1}{6} \left[ 2 + \lambda \right] \pi + \frac{1}{6} \left[ U_2^E + U_2^A \right] + \frac{1}{3} \hat{U}_2^E \tag{A.119}
\]

\[
D_2^A = \frac{1}{3} \left[ 1 - \lambda \right] \pi + \frac{1}{6} \left[ U_2^E + U_2^A \right] - \frac{1}{6} \hat{U}_2^E \tag{A.120}
\]

\[
D_2^V = \frac{1}{6} \left[ 2 + \lambda \right] \pi - \frac{1}{3} \left[ U_2^E + U_2^A \right] - \frac{1}{6} \hat{U}_2^E \tag{A.121}
\]
The expected utilities from search, $U_2^A$, $U_2^E$, and $\hat{U}_2^E$, are then defined by

$$F \equiv U_2^A (r + \delta_2) + \sigma - a_2 D_2^V \left[ D_2^A - U_2^A \right] = 0 \quad (A.122)$$

$$G \equiv U_2^E (r + \delta_2) + \sigma - a_2 D_2^V \left[ D_2^E - U_2^E \right] = 0 \quad (A.123)$$

$$H \equiv \hat{U}_2^E (r + \delta_2) + \gamma \sigma - a_2 \hat{D}_2^V \left[ \hat{D}_2^E - \hat{U}_2^E \right] = 0, \quad (A.124)$$

where $a_2 = \phi_2^3 / \sigma_2^V$. Using $H$ we find that

$$\frac{d\hat{U}_2^E}{d\lambda} = \frac{\frac{1}{2} a_2 \pi \left[ \hat{D}_2^E - \hat{U}_2^E + \hat{D}_2^V \right]}{r + \delta_2 + \frac{1}{2} a_2 \left[ \hat{D}_2^E - \hat{U}_2^E + \hat{D}_2^V \right]} > 0 \quad (A.125)$$

$$\frac{d\hat{U}_2^E}{d\gamma} = -\frac{\sigma}{r + \delta_2 + \frac{1}{2} a_2 \left[ \hat{D}_2^E - \hat{U}_2^E + \hat{D}_2^V \right]} < 0 \quad (A.126)$$

Next we show that $dU_2^A / d\lambda < 0$. Using Cramer’s rule we get $dU_2^A / d\lambda = A / B$, where

$$A = \begin{vmatrix} -\frac{\partial F}{\partial \lambda} & \frac{\partial F}{\partial U_2^A} & \frac{\partial F}{\partial D_2^V} \\ -\frac{\partial G}{\partial \lambda} & \frac{\partial G}{\partial U_2^A} & \frac{\partial G}{\partial D_2^V} \\ -\frac{\partial H}{\partial \lambda} & \frac{\partial H}{\partial U_2^A} & \frac{\partial H}{\partial D_2^V} \end{vmatrix} \quad B = \begin{vmatrix} \frac{\partial F}{\partial \lambda} & \frac{\partial F}{\partial U_2^E} & \frac{\partial F}{\partial \hat{U}_2^E} \\ \frac{\partial G}{\partial \lambda} & \frac{\partial G}{\partial U_2^E} & \frac{\partial G}{\partial \hat{U}_2^E} \\ \frac{\partial H}{\partial \lambda} & \frac{\partial H}{\partial U_2^E} & \frac{\partial H}{\partial \hat{U}_2^E} \end{vmatrix} \quad (A.127)$$

Consider first the denominator $B$. Since $\partial H / \partial U_2^A = 0$ and $\partial H / \partial U_2^E = 0$, we can write $B$ as

$$B = \frac{\partial H}{\partial \hat{U}_2^E} \left[ \frac{\partial F}{\partial U_2^A} \frac{\partial G}{\partial U_2^E} - \frac{\partial F}{\partial \hat{U}_2^E} \frac{\partial G}{\partial U_2^A} \right], \quad (A.128)$$

where

$$\frac{\partial H}{\partial \hat{U}_2^E} = r + \delta_2 + \frac{1}{2} a_2 \left[ \hat{D}_2^E - \hat{U}_2^E + \hat{D}_2^V \right] > 0. \quad (A.129)$$

Thus, $B > 0$ if

$$\frac{\partial F}{\partial U_2^A} \frac{\partial G}{\partial U_2^E} > \frac{\partial F}{\partial \hat{U}_2^E} \frac{\partial G}{\partial U_2^A}, \quad (A.130)$$

which can be written as

$$\left[ r + \delta_2 + \frac{1}{6} a_2 \left[ 2 \left( D_2^A - U_2^A \right) + 5 D_2^V \right] \right] \left[ r + \delta_2 + \frac{1}{6} a_2 \left[ 2 \left( D_2^E - U_2^E \right) + 5 D_2^V \right] \right]$$

$$> \frac{1}{6} a_2 \left[ 2 \left( D_2^A - U_2^A \right) - D_2^V \right] \frac{1}{6} a_2 \left[ 2 \left( D_2^E - U_2^E \right) - D_2^V \right]. \quad (A.131)$$

Note that this condition holds for all $r + \delta_2 > 0$ if it holds for $r + \delta_2 = 0$. Setting $r + \delta_2 = 0$ we get

$$12 \left[ D_2^A - U_2^A \right] D_2^V + 12 D_2^V \left[ D_2^E - U_2^E \right] + 24 \left[ D_2^V \right]^2 > 0. \quad (A.132)$$
Note that $D_2^E > U_2^E$ and $D_2^A > U_2^A$. Thus, this condition is satisfied, so that $B > 0$. Next consider the numerator $A$. With $\partial H / \partial U_2^E = 0$ we can write $A$ as

$$A = \left[ -\frac{\partial F}{\partial \lambda} \frac{\partial G}{\partial U_2^E} + \frac{\partial F}{\partial U_2^E} \frac{\partial G}{\partial \lambda} \right] \frac{\partial H}{\partial U_2^E} - \frac{\partial H}{\partial \lambda} \left[ \frac{\partial F}{\partial U_2^E} \frac{\partial G}{\partial U_2^E} - \frac{\partial F}{\partial \lambda} \frac{\partial G}{\partial U_2^E} \right] \equiv X_1 \equiv X_2.$$ (A.133)

Recall that $\partial H / \partial \hat{U}_2^E > 0$. Moreover,

$$\frac{\partial H}{\partial \lambda} = -\frac{1}{2} \pi a_2 \left( \hat{D}_2^E - \hat{U}_2^E + \hat{D}_2^V \right) < 0.$$ (A.134)

Thus, $A < 0$ when $X_1 < 0$ and $X_2 < 0$. Note that $X_1 < 0$ if

$$\frac{\partial F}{\partial U_2^E} \frac{\partial G}{\partial \hat{U}_2^E} < \frac{\partial F}{\partial \lambda} \frac{\partial G}{\partial U_2^E},$$ (A.135)

which can be written as

$$\frac{1}{6} a_2 \left[ 2 \left( D_2^A - U_2^A \right) - D_2^V \right] \frac{1}{6} \pi a_2 \left[ D_2^E - U_2^E + D_2^V \right] > \frac{1}{6} a_2 \left( D_2^A - U_2^A - 2D_2^V \right) \left[ r + \delta_2 + \frac{1}{6} a_2 \left[ 2 \left( D_2^E - U_2^E \right) + 5D_2^V \right] \right].$$ (A.136)

Simplifying yields

$$\frac{2}{a_2} \left( r + \delta_2 \right) \left[ \left( D_2^A - U_2^A \right) - 2D_2^V \right] + \left( D_2^A - U_2^A \right) \left( D_2^V - D_2^E - U_2^E \right) < 3 \left( D_2^V \right)^2.$$ (A.137)

From $F$ and $G$ we know that

$$D_2^V \left( D_2^A - U_2^A \right) = \frac{U_2^A \left( r + \delta_2 \right) + \sigma}{a_2} \quad \text{and} \quad D_2^V \left( D_2^E - U_2^E \right) = \frac{U_2^E \left( r + \delta_2 \right) + \sigma}{a_2},$$ (A.138)

so that condition (A.137) can be written as

$$\left( r + \delta_2 \right) \left( 2D_2^A - U_2^A - 4D_2^V - U_2^E \right) \equiv T < 3 \left( D_2^V \right)^2 a_2.$$ (A.139)

It remains to prove that $T < 0$. Using the definitions of $D_2^A$ and $D_2^V$ we can write $T < 0$ as

$$U_2^E + U_2^A < [1 + 2\lambda] \pi.$$ (A.140)

This condition is satisfied for all $\lambda \geq 0$ as $\pi > U_2^E + U_2^A$. Thus, $X_1 < 0$. Moreover, $X_2 < 0$ if

$$\frac{\partial F}{\partial U_2^E} \frac{\partial G}{\partial \hat{U}_2^E} < \frac{\partial F}{\partial \hat{U}_2^E} \frac{\partial G}{\partial U_2^E},$$ (A.141)
which is equivalent to
\[
\frac{1}{6}a_2\left[2\left(D_2^A - U_2^A\right) - D_2^V\right] - \frac{1}{6}a_2\left[D_2^E - U_2^E - 2D_2^V\right] < \frac{1}{6}a_2\left[D_2^A - U_2^A + D_2^V\right]\left[r + \delta_2 + \frac{1}{6}a_2\left[2\left(D_2^E - U_2^E\right) + 5D_2^V\right]\right].
\]

(A.142)

Again, \(D_2^A > U_2^A\) and \(D_2^E > U_2^E\). Thus, if this condition holds for \(r + \delta_2 = 0\), then it also holds for all \(r + \delta_2 > 0\). Setting \(r + \delta_2 = 0\) we get
\[
0 < 3D_2^V\left[D_2^E - U_2^E\right] + 9D_2^V\left[D_2^A - U_2^A\right] + 3D_2^V D_2^V. \tag{A.143}
\]

Hence, \(X_2 < 0\), so that \(A < 0\). Consequently, \(dU_2^E/d\lambda < 0\). Moreover, note that \(\partial D_2^E/\partial \lambda = \pi/6 < |\partial D_2^A/\partial \lambda| = \pi/3\). Thus, \(d\left[U_2^E + U_2^A\right]/d\lambda < 0\). Finally, using \(H\) we get
\[
\frac{d\hat{U}_2^E}{d\lambda} = \pi - \frac{\frac{1}{2}a_2\left[\hat{D}_2^E - \hat{U}_2^E + \hat{D}_2^V\right]}{r + \delta_2 + \frac{1}{2}a_2\left[\hat{D}_2^E - \hat{U}_2^E + \hat{D}_2^V\right]} = Z, \tag{A.144}
\]

where \(Z \in (0, 1)\). Thus,
\[
\frac{dD_2^Y}{d\lambda} = \frac{1}{6}a_2\left[1 - Z\right] - \frac{1}{3}d\left[U_2^E + U_2^A\right]. \tag{A.145}
\]

Consequently, \(dD_2^Y/d\lambda > 0\).

All this implies that the results from Proposition 5 continue to hold when the entrepreneur can search for a VC without the angel.

**Proof of Proposition 6.**

Recall that \(U_2^E = U_2^A\) in equilibrium. Moreover, as shown in Proof of Proposition 3, \(dU_2^E/d\phi_2 > 0\), and \(dU_2^E/d\sigma_2, dU_2^E/d\delta_2, dU_2^E/d\sigma_2^Y, dU_2^E/dk_2 < 0\). Consequently, \(d\gamma^*/d\phi_2 < 0\), and \(d\gamma^*/d\sigma_2, d\gamma^*/d\delta_2, d\gamma^*/d\sigma_2^Y, d\gamma^*/dk_2 > 0\). \(\Box\)

**Proof of Proposition 7.**

Recall from Proof of Proposition 5 that \(d\left[U_2^E + U_2^A\right]/d\lambda < 0\). Thus, \(d\gamma^*/d\lambda > 0\). \(\Box\)