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Heterogeneous Beliefs About Rare Event Risk in the Lucas Orchard

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Abstract

This paper investigates the asset pricing implications of investor disagreement about the likelihood of a systematic disaster. I specify a general equilibrium model with multiple trees and heterogeneous beliefs about rare event risk, to understand how risk-sharing mechanisms affect equity and variance risk premia, at an aggregate level and in the cross-section of stock returns. I identify a state-dependent link between equity and variance premia, that changes with the distribution of agent consumption. Empirically, as in the model, the variance premium’s predictive power for future excess returns is greater during times of financial distress, mainly for small stocks.

Keywords: heterogeneous beliefs, systematic disasters, Lucas orchard, variance risk premium, correlation risk premium, predictability

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1 Introduction

Since the outbreak of the global financial crisis in 2008, tail or disaster risk—one understood as the potential presence of infrequent adverse events of extreme magnitude—has been a concern for academics and investors alike. For instance, Hoang Le Huy, head of fixed income and event strategies at Schroders NewFinance Capital, a London-based fund of funds states: “You need to hedge against disaster scenarios. Black swan events are at the forefront for a lot of investors right now. It is not something that people take lightly. A lot of tail risk funds were built on the back of the 2008 disaster.” 1 Numerous studies show that even a small probability of an extreme event in economic fundamentals can have significant effects on asset prices. These extreme events are rare by definition and so accurately estimating their likelihood is difficult, which is a natural source of investor disagreement over perceived tail risk. Such heterogeneity of beliefs about disasters suggests that belief-driven risk sharing could explain the pricing of rare event risk. Compensation for disaster risk contributes to a significant fraction of expected returns on equity and pure variance positions. Thus, a better understanding of disaster risk premia should help explain the dynamics of equity and variance risk premia and the nature of their comovement. This paper studies, both theoretically and empirically, how agent disagreement about disaster risk affects excess return dynamics and the relation between the equity and the variance risk premia both for the market portfolio and the cross section of stocks.

I develop a general equilibrium Lucas (1978) economy with multiple assets and heterogeneous beliefs in which the premia for equity and variance positions and their comovement are endogenously driven by investor disagreement and the cross-sectional distribution of consumption. The model suggests a stronger predictive power of variance risk premium for future excess returns in periods during which pessimists have a relatively large consumption share—that is, in bad states of the economy, which are also characterized by higher (absolute) values of the variance risk premium. Accordingly, I find empirically that variance risk premia and their predictive power for future excess returns are concentrated in phases of substantial disagreement among investors. In these phases, regression coefficients and $R^2$ are particularly large for small stocks, whose re-

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1See Risk.net.
turns are more dependent on the compensation for systematic rare event risk. Therefore, exposure to aggregate variance risk could partially explain the size effect (i.e. the observation that smaller firms have higher returns on average), which actually seems to be most pronounced during periods in which pessimists hold a large fraction of the aggregate endowment. The time variation in the sign and strength of the predictive power of the aggregate variance premium—both for future excess market returns and for individual stocks—is a challenge for existing consumption-based asset pricing models. The model I posit addresses these empirical challenges, and I provide a structural explanation based on the role of risk sharing between agents who disagree.

The main ingredients of the model are the following. First, I consider an endowment economy with a single consumption good but multiple trees.\(^2\) The endowment processes follow a geometric Brownian motion with the addition of idiosyncratic and systematic jump components. The presence of multiple trees allows me to study the relation between equity premia in the cross section and the aggregate variance premium, together with the determinants of comovement between trees. Second, two groups of investors, who have constant relative risk aversion (CRRA) preferences over consumption, have different beliefs about the likelihood of a rare systematic event. Disagreement is an important source of non continuous variation in the variance risk premium dynamics. The observed market variance premium is, in fact, highly time varying; periods of small and smooth premium alternate with periods in which the variance premium is larger (in absolute value) and more volatile. The presence of disagreement about the intensity of disasters also allows the variance risk premium to switch sign in certain phases, mainly for small individual stocks; such switching is consistent with the empirical evidence. Third, I assume that the intensity of the systematic jump process is time varying and proportional to an exogenous state variable that can be interpreted as a continuous signal reflecting the state of the economy. The two agents disagree on the coefficient of proportionality, so that the absolute difference in perceived expected growth rates is also proportional to the exogenous state variable and never switches sign. This simple specification of disagreement can be considered as a reduced-form way to capture several empirical regularities of differences in opinion which have been recently documented.

\(^2\)In this context, trees are assets and a collection of trees is an orchard. See e.g. Martin (2013).
Borrowing from the solution methods proposed for the Lucas Orchard by Martin (2013) and from methods used in the single-asset difference in beliefs model of Chen, Joslin, and Tran (2012), I derive semi-closed-form expressions for the stock prices in my multiple trees economy with heterogeneous beliefs. Price-dividend ratios of individual stocks and parameters in the price process dynamics depend on the consumption share of the two agents, on the state variable driving time-varying intensities, and on the dividend share distribution.

Using the model solution, I derive a number of predictions. First, the equity (variance) risk premium of an individual stock tends to increase (decrease) with its dividend share and with the consumption share of the pessimistic agent; these phenomena can be explained by the risk-sharing behavior of disagreeing investors. Moreover, as noted above, the variance risk premium can switch sign—in particular for small stocks—when optimists consume a large fraction of the aggregate endowment and disagreement is large enough. In line with the data, the variance premium is time varying; it alternates phases of small and smooth premia with periods in which the variance premium is larger (in absolute value) and more volatile, where the change in regime is driven by an abrupt change in the cross-sectional distribution of agent consumption. Second, the model-implied correlation risk premium inherits these features because, consistently with the empirical findings of Driessen, Maenhout, and Vilkov (2012), the index variance risk premium is largely due to a covariance premium, mainly when assets are relatively evenly distributed or the number of stocks in the economy is large. While the cross-sectional distribution of agent consumption mainly affects the risk-neutral stock return correlation, the physical correlation is relatively insensitive to it, which leads to a countercyclical correlation risk premium. Third, rare event risk implies a tight link between the equity and the variance risk premia, both for the market and for the cross section of stock returns. This link provides the basic intuition for the role of the variance premium in predicting future excess returns. However, standard predictive regressions imply an unconditionally linear relation between equity and variance risk premia, whereas in the model the regression coefficients are stochastic and depend on the asset’s dividend share and the agents’ consumption share. I show by simulation that the aggregate variance premium’s power to predict future excess returns is stronger when the consumption share of the pessimist is
larger, i.e., in bad states of the economy. At a disaggregate level, the predictive power of the variance risk premium is especially large for small stocks. Fourth, I consider the special case of a large diversified economy in which the number of stocks approaches infinity and all stocks have the same dividend share, which approaches zero. In this case, only systematic risk is priced and the relation between equity and variance risk premia is conditionally linear. Moreover, infinitely small assets still earn a risk premium owing to the presence of systematic rare event risk.

The empirical validity of the model’s main predictions is studied using the aggregate S&P 500 composite index as a proxy for the aggregate market and the return time series of CRSP cap-based portfolio returns to analyze the differential effects of small versus big stocks, based on monthly data from January 1990 through December 2011. The empirical evidence confirms that the index variance premium’s ability to predict future excess returns is (a) time varying for the market and for single stocks or stock portfolios and (b) stronger during periods of financial distress. Such periods are characterized by large (absolute) variance premia and substantial investor disagreement, which is proxied by the dispersion in one-year-ahead forecasts of real GDP growth from the BlueChip Economic Indicator. The predictive power of the variance premium is stronger (on average) for small stocks, which have returns that depend more on the compensation for systematic rare event risk. For example, the adjusted $R^2$ of a standard predictive regression of excess six-month returns on the aggregate variance premium is about 63% larger for the small-cap portfolio than for big caps. The difference between small- and big-cap portfolios is particularly evident in periods of high disagreement. Intuitively, investors will require higher return from assets that are more sensitive to systematic disaster risk. However, this reasoning holds only when the perceived premium for systematic jumps is sufficiently large. The model suggests that the systematic jump premium component can even have a negative effect on a stock’s excess returns if the pessimists’ consumption share is low enough. Thus the size premium could move in opposite directions depending on what agent type dominates the market. This finding is consistent with the mixed

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3The empirical literature on the size premium identifies several reasons why small stocks are more sensitive to systematic risk. One possible explanation is that small firms are more affected by tight credit conditions.
results reported in the empirical literature on size premia.

This paper is related to several different strands of the literature. The first is the growing research on asset pricing with multiple trees. Cochrane, Longstaff, and Santa-Clara (2008) highlight the asset pricing implications of a two-trees Lucas (1978) economy with a log-utility representative investor. Martin (2013) introduces multiple Lucas trees (Lucas orchard) following jump-diffusion processes and a representative agent with power utility. Buraschi, Trojani, and Vedolin (2014b) specify a diffusive two-trees model with heterogeneity in beliefs; they characterize the relation between the difference in opinions, volatility and correlation risk premia of index and individual options. In contrast to previous papers, I specify a collection of Lucas trees with rare disasters and heterogeneous beliefs about the intensity of systematic rare events, with the goal of studying the implications for equilibrium risk premia and for the relation between the market variance risk premia and excess returns. Multiple trees allow me to analyze the contribution of a premium for covariance risk to this predictive relation. This insight is motivated by the empirical evidence in Driessen, Maenhout, and Vilkov (2012) that the index variance risk premium is largely due to the high price of correlation risk and that option-implied correlations have remarkable predictive power for future stock market returns. In my model, covariance risk can contribute to a large fraction of the aggregate variance premium when the economy is dominated by pessimistic agents. In such states, fear of systematic disasters requires large compensation for both equity and variance risks, leading to a strong comovement between equity and variance risk premia.

The predictive relation between the market variance risk premium and excess returns was first observed by Bollerslev, Tauchen, and Zhou (2009), who provide empirical evidence that the variance risk premium accounts for a nontrivial fraction of the time-series variation in post-1990 aggregate stock market returns at short horizons. They motivate this link theoretically in a long-run risk model with stochastic volatility of consumption growth volatility. In such models, recursive preferences are crucial to generate a premium for stochastic volatility of consumption growth volatility, which then drives both the equity and variance risk premia. Yet, Wu (2012) observes that there is no empirical correlation between the variance risk premium and the volatility of consumption growth volatility. Moreover, Bollerslev and Todorov (2011) find that more than half of
the variance risk premium is driven by disaster risk and suggest that equilibrium-based asset pricing models should accommodate large and time-varying compensation for rare disasters. The idea that the possibility of sudden downward jumps in the endowment may help explain the equity premium puzzle dates back to Rietz (1988). More recently, Gabaix (2012) and Wachter (2013) resolve several asset pricing puzzles by including a time-varying risk of disasters in otherwise standard models. The literature on rare disasters does not seek to explain the variance risk premium’s puzzling dynamics or its ability to predict future excess returns. This paper seeks to fill that gap starting from a general equilibrium model in which two sets of agents have different beliefs about the probability of a disaster occurring.

Previous papers have studied the disagreement that surrounds assessments of disaster risk. Dieckmann (2011) provides an equilibrium model in which log-utility investors have heterogeneous beliefs about the likelihood of rare events; he explores the asset pricing implications of this setup in an incomplete capital market as well as the effects of market completion. Chen, Joslin, and Tran (2012) consider a complete market setting and assume that two CRRA agents disagree about rare event risk. They show that the relation between the disaster risk premium and the extent of disagreement about disaster risk is highly nonlinear; a small proportion of optimistic investors can greatly attenuate the impact of disaster risk on stock prices. I contribute to this literature along several dimensions. First, I study the effects of disagreement on variance risk premia and its predictive power for excess returns. Second, using the multiple trees setting I study the cross-sectional implications of heterogeneous rare event risk. Third, I test empirically the model’s main predictions. Finally, I use a specification of disagreement that is consistent with several empirical regularities of differences in opinion. Patton and Timmermann (2010) and Buraschi, Trojani, and Vedolin (2014a) show that differences in beliefs are highly time varying and countercyclical. Moreover, Patton and Timmermann (2010) suggest that there is a strong negative correlation between dispersion and consensus forecast on GDP growth. They also find that forecasters’ view are persistent—in other words, they tend to be consistently optimistic or pessimistic. In my model, disagreement is countercyclical whereas the average belief about expected consumption growth is procyclical; these dynamics lead to a perfect negative correlation between consensus and dispersion,
whose persistence is guaranteed by a positive exogenous state variable. Finally, my results on the connection between the cross section of excess stock returns and the aggregate variance premium are related to the literature on the size effect. Lemmon and Portniaguina (2006) demonstrate a negative relation between the size premium and consumer confidence. In fact, the size effect (whereby smaller firms have higher returns on average) seems to be concentrated in periods characterized by large disagreement. Intuitively, investors tend to require higher returns from assets that are more sensitive to systematic disaster risk. This intuition is well explained by Jagannathan and Wang (1996), who argue that the market beta of firms with a greater likelihood of financial distress (e.g., small firms) is more sensitive to changes in the business cycle. Investor sentiment is thus related to time variation in the expected returns of those firms because such sentiment forecasts future business conditions. However, this reasoning holds only when the perceived systematic jump premium is high. My model indicates that the jump premium component can actually have a negative effect on the stock’s excess returns if the consumption share of the pessimists is sufficiently low. Therefore, the size premium can move in opposite directions, depending on which agent type dominates the market, consistently with the mixed results of the later empirical research on the size effect.

The rest of the paper is organized as follows. Section 2 introduces the basic model setup as well as the optimal consumption allocation, market prices of risk, and equilibrium market prices. Section 3 analyzes the properties of the equity and variance risk premia, of their relationship, and the correlation risk premium. It also studies the case of an infinitely large and diversified economy. Section 4 describes the data and provides empirical support for the main model implications, while Section 5 concludes. All proofs can be found in the Supplemental Appendix.

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4See e.g. Crain (2011) for a review of the size effect.
2 An Economy with Multiple Trees and Heterogeneous Beliefs about Systematic Disasters

This section introduces the model, which is a simple, continuous-time generalization of the standard Lucas (1978) endowment economy. The model incorporates rare disasters and heterogeneous beliefs about the probability of a common jump in $N$ Lucas trees. For notational convenience, vectors and matrices are denoted by bold symbols.

Two agents ($i = A, B$) observe the dividend stream produced by each tree, $D_j$, with the following exogenous dynamics:

$$
\frac{dD_j(t)}{D_j(t)} = \mu_j dt + \sigma_j dW_{jt} + k_j dN_{jt} + k_j dN_{ct}, \quad j = 1, \ldots, N, \quad (1)
$$

where $W_t = (W_{1t}, W_{2t}, \ldots, W_{Nt})'$ is an $N$-dimensional standard Brownian motion driving regular economic risk, rare event risk enters through the Poisson processes $N_t = (N_{1t}, N_{2t}, \ldots, N_{Nt})'$ and $N_{ct}$ with respective intensities $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_N)'$ and $\lambda_c(t)$. Thus each stock has both an idiosyncratic and a systematic event risk component. Namely, the jump in the dividend growth of stock $j$ is idiosyncratic if driven by a jump in $N_{jt}$ whereas jumps in $N_{ct}$ are common to all stocks. For simplicity, and since the goal is to understand the asset pricing implications of heterogeneous beliefs on the probability of a common jump, I assume that the intensity of the systematic Poisson process is time varying\(^5\) while all other parameters, including the idiosyncratic jump intensities, are constant. Furthermore, the coefficients $\mu_j, \sigma_j, k_j$, and $\lambda_j$—which represent, respectively, the expected growth rate and volatility of dividend growth without jumps, the jump size, and the idiosyncratic jump intensity—are assumed to be identical for all trees in the economy (hence I will suppress their subscript $j$). The jump size $k$ is restricted to be negative and strictly less than 1 in absolute value; this ensures that dividend processes are positive.\(^6\)

To focus on the effects of heterogeneous systematic rare event risk on risk premia, I

\(^5\)Wachter (2013), among others, underlies the importance of taking into account time variation in the probability of rare disasters to help explain, e.g., time variation in the equity premium and the excess volatility puzzle, while Berkman, Jacobsen, and Lee (2011) provide empirical support for time-varying rare disaster intensity.

\(^6\)The assumption of constant jump size could be relaxed, but it helps to maintain tractability and to isolate the effect of disagreement about rare event intensity.
assume that agent beliefs differ only with respect to the systematic rare event intensity $\lambda_c(t)$, which is a function of an exogenous affine state variable $X(t)$. In particular, agent $i$ believes that the common jump frequency is given by

$$\lambda^i_c(t) = \beta^i X(t), \quad (2)$$

for $i = A, B$, where $X(t)$ follows a CIR process,

$$dX(t) = \varphi [1 - X(t)] dt + \sigma_X \sqrt{X(t)} dW^X_t, \quad (3)$$

for $W^X_t$ a standard Brownian motion that is independent of $W_t$. This assumption ensures positivity of the intensity of a common jump under each agent’s beliefs, which also follows a CIR process.\(^8\) I assume that the long-term mean of $X$ is equal to 1, so that $\beta^i$ represents the expected systematic rare event intensity perceived by agent $i$.

The probability measures of the two agents are equivalent because they agree on null sets. Hence, their Radon–Nikodym derivative $\phi(t) = d\mathbb{P}^B / d\mathbb{P}^A$ exists and has the following dynamics (see Chen, Joslin, and Tran (2010)):

$$\frac{d\phi(t)}{\phi(t)} = (\beta^A - \beta^B) X(t) dt + \left[ \frac{\beta^B}{\beta^A} - 1 \right] dN_{ct}. \quad (4)$$

If the agents observe a common jump (i.e. if $dN_{ct} = 1$) then the likelihood ratio jumps by a factor of $\beta^B / \beta^A$. If agent $B$ is optimistic—which means he believes that the probability of a systematic (negative) jump is lower (i.e., $\beta^B < \beta^A$, as I assume throughout the paper)—then $\phi$ jumps downward in response to a systematic disaster. The absence of systematic jumps over a period of time instead is more consistent with the optimist’s beliefs and so the likelihood ratio increases deterministically at a rate $\lambda^A_c - \lambda^B_c$. Note that, even in the case of constant systematic rare event intensity (i.e., $X$ constant) the state variable $\phi$ varies over time and decreases dramatically following a disaster.

\(^7\)Benzoni, Collin-Dufresne, Goldstein, and Helwege (2012) assume a similar dynamics for a country’s default intensity and use a $\beta$ parameter that depends on the state of the world. They then employ learning to capture contagion effects in the perceived default intensities of different countries.

\(^8\)Wachter (2013) and Chen, Joslin, and Tran (2012) also assume CIR processes for the rare event intensity. Chen, Joslin, and Tran (2012) include disagreement directly in the long-run average jump intensity whereas here the proportionality coefficient $\beta^i$ is used to introduce disagreement.
2.1 Dividend shares and consumption dynamics

I consider an endowment economy in which all trees produce the same perishable consumption good; therefore, aggregate consumption equals the sum of all dividends, i.e. $C(t) = \sum_{j=1}^{N} D_j(t)$. Let $s_j$ be the share of consumption contributed by stock $j$, i.e. $s_j = \frac{D_j(t)}{C(t)}$. An application of Itô’s lemma to Equation (1) gives its dynamics:

$$ds_j = \sigma^2 s_j \left( \sum_{i=1}^{N} s_i^2 - s_j \right) dt + \sigma s_j \left( dW_{jt} - \sum_{i=1}^{N} s_idW_{it} \right) + \frac{k}{ks_j + 1} s_j dN_{jt} - s_j \sum_{i \neq j} \frac{k s_i}{ks_i + 1} dN_{it}. \tag{5}$$

Intuitively, the dividend share of asset $j$ increases when there is a positive Brownian shock to its dividend growth dynamics or an idiosyncratic disaster involving any of the other dividend processes. Systematic jumps do not affect dividend share dynamics because such jumps are assumed to have the same impact on all dividend processes. By construction, the dividend shares $s_j$ sum to 1 and the $N - 1$ state variables $s_j$ for $j = 2, \ldots, N$ are enough to describe the relative size of the $N$ trees.\(^9\)

From 1, the dynamics of aggregate consumption growth is given by

$$\frac{dC(t)}{C(t)} = \mu dt + \sigma \sum_{j=1}^{N} s_j dW_{jt} + k \sum_{j=1}^{N} s_j dN_{jt} + kdN_{ct}. \tag{6}$$

Observe that even if agents agree on $\mu$, the growth rate of consumption in normal times, disagreement about the systematic rare event intensity leads to disagreement about the total expected growth rate,

$$\mu^i_C = E^i_t \left[ \frac{dC(t)}{C(t)} \right] = \mu + k(\lambda^i_t(t)). \tag{7}$$

Here $E^i_t(.)$ denotes conditional expectation under the probability measure $\mathbb{P}^i$, which summarizes agent $i$’s beliefs. Thus, the difference in expected growth rates can be expressed in terms of the dispersion in beliefs,

$$\mu^B_C - \mu^A_C = k(\lambda^B_c(t) - \lambda^A_c(t)) = -k(\beta^A - \beta^B)X(t), \tag{8}$$

\(^9\)The drift in Equation (5) is zero when $s_j = 0$, $1/N$, or 1, and the dividend share distribution is not stationary because one asset ultimately becomes dominant in the market; that is, $ds_j = 0$ for $s_j = 0, 1$. This feature is shared by many of the literature’s general equilibrium models that involve two or more trees. See for example Cochrane, Longstaff, and Santa-Clara (2008), who discuss properties of the dividend share dynamics in the case of two trees.
which is linear in the exogenous state variable $X$.

This simple specification of disagreement is a parsimonious way to capture several empirical regularities of differences in opinion that have been reported recently. Patton and Timmermann (2010) and Buraschi, Trojani, and Vedolin (2014a) show that differences in beliefs are highly time varying and countercyclical. Moreover, Patton and Timmermann (2010) suggest that (a) there is a strong negative correlation between belief dispersion and a consensus forecast of GDP growth and (b) forecasters’ views tend to be consistently optimistic or consistently pessimistic. In the model, disagreement is countercyclical if the state variable $X$ is interpreted as an exogenous continuous signal about the state of the economy. The average belief as regards expected consumption growth is a decreasing function of $X$, while the absolute difference in perceived expected growth rates is increasing in $X$; the result is a perfect negative correlation between consensus forecast and the dispersion in forecasters’ beliefs, the persistence of which is guaranteed by the positivity of $X$.\footnote{Disagreement could instead switch sign in the single-asset belief disagreement model of Chen, Joslin, and Tran (2012), even if disaster intensities follow CIR processes, because disagreement is introduced directly in the long-run average jump intensity.}

### 2.2 Agent optimization problem

Agents have a constant relative risk aversion (CRRA) utility over consumption with finite horizon $T$:

$$ U^i(C^i(t)) = \frac{C^i(t)^{1-\gamma}}{1 - \gamma} $$

for $i = A, B$; here $\gamma$ is the coefficient of relative risk aversion, which is assumed to be identical across agents.\footnote{Dieckmann and Gallmeyer (2005) introduce heterogeneity only through different levels of relative risk aversion and study equilibrium allocations. Chabakauri (2013) considers two trees and two CRRA investors with heterogeneous risk aversions and portfolio constraints, and he examines the effects on return correlations and volatilities. Chen, Joslin, and Tran (2012) argue that combining heterogeneous beliefs about disasters and different risk aversions can amplify the effects of risk sharing but does not qualitatively change basic asset pricing results.} If we assume complete markets and use martingale techniques (see e.g. Cox and Huang (1989)) then agent $i$’s optimization problem can be written in
static form as

\[ J^i = \max_{C^i} E^i \left[ \int_0^T e^{-\delta t} U^i(C^i(t)) \, dt \right], \quad \text{s.t.} \quad E^i \left[ \int_0^T \eta^i(t) C^i(t) \, dt \right] \leq W^i(0). \tag{10} \]

Here \( \delta \) is the time preference rate and \( \eta^i(t) \) is the state price density of agent \( i \), whose dynamics is given by

\[
\frac{d\eta^i(t)}{\eta^i(t)} = -r(t) dt + \sum_{j=1}^N \left( \lambda^Q_j(t) - \lambda^Q_i(t) \right) dt + \sum_{j=1}^N \left( \frac{\lambda^Q_j(t)}{\lambda^Q_i(t)} - 1 \right) dN_{jt} + \theta(t)' dW_t + \sum_{j=1}^N \left( \lambda^Q_j(t) - 1 \right) dN_{ct}. \tag{11} \]

In this expression, the \( N \)-vector \( \theta \) is the market price of regular economic risk associated with Brownian motion \( W(t) \), while \( \lambda^Q_j(t) \) and \( \lambda^Q_i(t) \) are the risk-neutral rare event intensities associated with the respective Poisson processes \( N_j(t) \) and \( N_c(t) \).\(^\text{12}\) Agents are assumed to be initially endowed with a fraction \( x^i_s \) of each stock; that is, \( W^i(0) = x^i_s \sum_{j=1}^N S_j(0) \).

The standard optimality condition now yields

\[ C^i(t) = I^i \left( y^i \eta^i(t) e^{\delta t} \right) = \left( y^i \eta^i(t) e^{\delta t} \right)^{-1/\gamma}, \]

where \( I^i(\cdot) \) is the inverse marginal utility function of agent \( i \) and \( y^i \) is the Lagrange multiplier that solves the following budget constraint:

\[ E^i \left[ \int_0^T \eta^i(t) I^i \left( y^i \eta^i(t) e^{\delta t} \right) dt \right] = W^i(0). \]

The equilibrium allocations can be characterized by solving the optimization problem of a representative agent whose utility function is a weighted sum of the two agents’ utilities. Hence the planner’s problem (under \( P^A \), the pessimist’s probability measure) is as follows:

\[ J = \max_{C^A, C^B} E^A \left[ \int_0^T e^{-\delta t} \left( \frac{C^A(t)^{1-\gamma}}{1-\gamma} + \phi(t) \frac{C^B(t)^{1-\gamma}}{1-\gamma} \right) dt \right], \quad \text{s.t.} \quad C^A(t) + C^B(t) = C(t), \tag{12} \]

where the weight \( \phi \) is stochastic and is driven by the difference in beliefs.\(^\text{13}\) The equilibrium consumption allocations are obtained from the first-order condition of the representative agent problem with state-dependent weight was introduced by Cuoco and He (1994); more recent examples can be found in Basak and Cuoco (1998) and Buraschi and Jiltsov (2006). In a complete markets setting with heterogeneous beliefs the weight is stochastic and equal to the Radon–Nikodym derivative \( \phi(t) = dP^B/dP^A \).

\(^\text{12}\)The market prices of diffusion and jump risk are not agent specific if the market is complete, since the agents have to agree on the observed price paths, see e.g. Dieckmann (2011). Completeness of the market is discussed in Section 2.3.

\(^\text{13}\)The approach to formulating a representative agent problem with state-dependent weight was introduced by Cuoco and He (1994); more recent examples can be found in Basak and Cuoco (1998) and Buraschi and Jiltsov (2006). In a complete markets setting with heterogeneous beliefs the weight is stochastic and equal to the Radon–Nikodym derivative \( \phi(t) = dP^B/dP^A \).
tative agent’s problem while using individual agents’ optimality conditions.

**Proposition 1** Equilibrium consumption allocations are

\[
C^A(t) = \frac{1}{1 + \phi(t)^{1/\gamma}} C(t) \quad \text{and} \quad C^B(t) = \frac{\phi(t)^{1/\gamma}}{1 + \phi(t)^{1/\gamma}} C(t),
\]

and investors’ state price densities are

\[
\eta^A(t) = e^{-\delta t} \left( 1 + \phi(t)^{1/\gamma} \right)^\gamma y^A C(t)^\gamma \quad \text{and} \quad \eta^B(t) = \eta^A(t) \frac{\phi(0)}{\phi(t)} = e^{-\delta t} \left( 1 + \phi(t)^{1/\gamma} \right)^\gamma y^B C(t)^\gamma \phi(t).
\]

Here \( \phi(0) \) solves either agent’s individual budget constraint,\(^{14}\) and the stochastic weighting process \( \phi(t) = y^A \eta^A(t)/y^B \eta^B(t) \) follows the dynamics given in Equation (4) with jump intensity \( \lambda_c^A(t) \).

Proposition 1 characterizes the dependence of individual state price densities and consumption policies on \( C \) and \( \phi \), which represent aggregate endowment and belief disagreement risk in the economy, respectively. In this \( N \)-trees setting, the aggregate endowment \( C \) depends on the exogenous single dividend growth processes and on the dividend shares. Under homogeneous beliefs, the disagreement risk vanishes because \( \phi \) is constant and depends only on initial wealth: \( \phi = (x_s^B/x_s^A)^\gamma \). This means that, under homogeneous beliefs, the investors who are initially more wealthy consume more in all future states and times. In contrast, consumption differences can change sign if agents are heterogeneous. Namely, if a systematic disaster occurs, the consumption share of the pessimist (agent \( A \)) increases as \( \phi \) jumps down.

For convenience, I define explicitly the consumption shares of the pessimistic and optimistic agents as

\[
c^A(t) \equiv \frac{C^A(t)}{C(t)} \quad \text{and} \quad c^B(t) \equiv \frac{C^B(t)}{C(t)},
\]

respectively. These shares will drive the market prices of risk and risk premia.

### 2.3 Price processes and market completeness

Assume the existence of a capital market that allows agents to share risk and finance consumption. The market consists of \( N \) risky assets with price vector \( S(t) = (S_1(t), S_2(t), \ldots, S_N(t))^\prime \),

---

\(^{14}\)The budget constraints of agents determine only the ratio \( y^A/y^B \). I set \( y^A = U'(C(0), \phi(0)) \) without loss of generality, so that \( \eta^A(0) = \eta^B(0) = 1 \).
each in unit net supply, as well as a riskless asset of price $B(t)$ in zero net supply. Then, for $j = 1, \ldots, N$, the price process dynamics are as follows:

$$
\frac{dS_j(t) + D_j(t)dt}{S_j(t)} = \mu_{S_j}(t)dt + \sigma_{S_j}(t)[dW_t' \cdot dW_t^X] + k_{S_j}(t)dN_t + k_{S_j}^c(t)dN_{ct},
$$

$$
\frac{dB(t)}{B(t)} = r(t)dt.
$$

The expected returns in normal times, $\mu_{S_j}$, the $1 \times N + 1$ vectors $\sigma_{S_j}$ of diffusion volatilities, the $1 \times N$ vectors $k_{S_j}$ of return jump sizes related to idiosyncratic jumps, the return jump sizes related to systematic jumps, $k_{S_j}^c$, and the riskless rate $r$, are determined endogenously in equilibrium. However, with only $N$ risky securities the market is incomplete, since they only span the uncertainty driven by the Brownian motions.$^{15}$ Hence I assume agents can also trade in $N + 1$ rare event insurance products $P_j(t), j = 1, \ldots, N$ and $P_c(t)$, which are in zero-net supply, do not pay dividends, and have price processes

$$
\frac{dP_j(t)}{P_j(t)} = \mu_{p_j}(t)dt + k_{p_j}(t)dN_{jt},
$$

$$
\frac{dP_c(t)}{P_c(t)} = \mu_{p_c}(t)dt + k_{p_c}(t)dN_{ct},
$$

where $\mu_{p_j}$ and $\mu_{p_c}$ are determined in equilibrium, whereas jump sizes can be freely chosen and need only be different from zero in order to complete the market. These assets can be interpreted as insurance products against rare event risk because they do not contain any continuous source of uncertainty. The buyer of asset $P_j, j = 1, \ldots, N, c$, is rewarded in the amount $\mu_{p_j}$ every moment of time, but runs the risk that the asset’s value drops to $(1 + k_{p_j})P_j$ when the corresponding Poisson process $N_{jt}$ jumps. Therefore, selling assets $P_j, j = 1, \ldots, N$, provides insurance against idiosyncratic jumps, while $P_c$ is a form of insurance against systematic disasters.$^{16}$ In general, any set of $N + 1$ assets spanning all jump components would complete the market, but the choice of disaster insurances is the most appealing since it isolates the impact of the different rare events.

$^{15}$More precisely, there are $N + 1$ Brownian shocks in the economy; however, the risk of changes in the disaster probability (i.e., shocks to $W_t^X$) are not priced in the power utility setting although they would be if agents had recursive preferences. See also Wachter (2013).

$^{16}$Catastrophe bonds can be viewed as the real-world counterpart to these theoretical securities.
2.4 Market prices of risk

Market prices of risk are obtained by applying Itô’s lemma to Equation (14) and then comparing the resulting dynamics with Equation (11). They are summarized in the following proposition.

Proposition 2 The market prices of normal economic risk, both risk-neutral rare event intensities, and the short rate are given by

\[
\begin{align*}
\theta_j(t) &= \gamma s_j \sigma, \\
\lambda^Q_j &= \lambda(s_j k + 1)^{-\gamma}, \\
\lambda^Q_c(t) &= (c^A(t)\lambda^A_c(t)^{1/\gamma} + c^B(t)\lambda^B_c(t)^{1/\gamma})^{-\gamma} (k + 1)^{-\gamma}, \\
r(t) &= \delta + \gamma \mu - \frac{1}{2} \gamma^2 (\gamma + 1) \sum_{j=1}^{N} s_j^2 \sigma^2 - c^B(t)(\lambda^A_c(t) - \lambda^B_c(t)) + \\
&\quad + \sum_{j=1}^{N} (\lambda - \lambda^Q_j(t)) + (\lambda^A_c(t) - \lambda^Q_c(t)).
\end{align*}
\]

The market price of economic risk has the standard solution as extended to the case of \(N\) trees. The risk-neutral intensity \(\lambda^Q_j\) of an idiosyncratic jump in the dividend process \(D_j\) depends only on the dividend share of asset \(j\) given that agents agree on the physical idiosyncratic jump intensities. For any dividend share distribution, the idiosyncratic jump risk premia \(\lambda^Q_j(t)/\lambda\) are constant and always greater than 1, and they tend to unity as \(s_j\) approaches zero. Thus the risk of idiosyncratic jumps in small assets is not priced, and in general the price associated with idiosyncratic jump risk is small when the number of stocks in the economy, \(N\), is large. The risk-neutral common disaster frequency \(\lambda^Q_c\) is a nonlinear function of the two agents common jump intensities weighted by their consumption shares; it could be smaller than the physical intensity when the optimist’s consumption share is large, leading to a systematic jump premium of less than 1. The riskless interest rate follows the standard expression in Lucas economies with multiple trees (see e.g. Cochrane, Longstaff, and Santa-Clara (2008)), with the addition of three components related to disagreement and jump premia. The equilibrium short rate is generally decreasing with the consumption share of the pessimistic agent \(A\), as in the aftermath of a disaster.
Using Proposition 2, it is possible to derive explicitly the risk premia on the risky assets once the volatilities and jump sizes of the stock price processes are known. This can be done by applying Itô’s lemma to the stock prices. For integer risk aversion γ, the resulting equation can be solved in semi-closed-form as summarized in the next proposition.

**Proposition 3** The price of stock \( j \) is given by

\[
S_j(t) = E_t^1 \left[ \int_t^T \frac{n_A(s)}{n_A(t)} D_j(s) \, ds \right] = D_j(t)g_j(\phi(t), X(t), u(t), t).
\]

(23)

Here the price-dividend ratio \( g_j \) depends on time \( t \), on the stochastic weighting process \( \phi \), on the state variable \( X \) that drives time-varying systematic disaster intensity, and on the dividend share distribution through the \((N - 1)\)-dimensional state variable \( u = (u_2, \ldots, u_N)' \), with \( u_j = \ln \frac{a_j}{\sigma_1} \):

\[
g_j(\phi, X, u, t) = e^{-\gamma \sum_{i=2}^N a_j/N} (1 + e^{u_2} + \cdots + e^{u_N}) \gamma \sum_{k=0}^\gamma a_k(\phi) \int \mathcal{F}_\gamma^N(z) e^{iu'z} b_{jk}(X, t, z) \, dz,
\]

where the integral is evaluated on \( \mathbb{R}^{N-1} \), \( \mathcal{F}_\gamma^N(z) \) is given in the Supplemental Appendix and

\[
a_k(\phi) = \binom{\gamma}{k} \frac{\phi(t)^{k/\gamma}}{(1 + \phi(t))^{3/\gamma}^\gamma},
\]

\[
b_{jk}(X, t, z) = \int_t^T e^{(t-\tau)\gamma} \left[ -\delta + (\mu - \frac{1}{2}\sigma^2)X_\gamma + \frac{1}{2}\sigma^2 X_\gamma \right] \alpha_{0,k}(\tau - t) + \alpha_{2,k}(\tau - t) \right] \, d\tau.
\]

Here \( e_j \) is the \( N \)-vector with a 1 in the \( j \)th entry and 0s elsewhere, \( 1_N \) is an \( N \)-dimensional vector of 1s and \( \alpha_{0,k}^N(\tau) \) and \( \alpha_{2,k}^N(\tau) \) satisfy the system of Riccati equations given in the Supplemental Appendix.

Semi-closed-form expressions\(^\text{17}\) for diffusion volatilities and jump sizes of stock \( j \)’s return process follow after application of Itô’s lemma for jump-diffusion processes, using dividend growth, stochastic weight process, dividend shares and exogenous state variable dynamics:

\[
\mathbf{\sigma}_{S_j}(t) = \left[ \sigma \left( e'_j + \frac{g'_j u}{g_j} U \right) g_j X \mathbf{\sigma}_X \sqrt{X(t)} \right], \quad k_{S_j}(t) = (k + 1) \frac{g_j \left( \frac{g'_j}{g_j}, X, u, t \right)}{g_j(\phi, X, u, t)} - 1.
\]

\(^{17}\)Up to the solution of the ordinary differential equations for \( \alpha_{0,k}^N(\tau) \) and \( \alpha_{2,k}^N(\tau) \), which is easily obtained numerically after evaluating an \((N - 1)\)-dimensional integral that is well-behaved but can be computationally intensive for large \( N \).
The $i$th component of vector $k_{S_j}(t)$ is given by

$$k_{S_j,i}(t) = \begin{cases} 
g_j(\phi,X,u+\ln(k+1)Ue_i,t) - 1 & \text{if } i \neq j, \\
(k + 1)^2 g_j(\phi,X,u+\ln(k+1)Ue_i,t) \frac{g_j(\phi,X,u,t)}{g_j(\phi,X,u,t)} - 1 & \text{if } i = j, 
\end{cases}$$

where $U$ is a $(N - 1) \times N$ matrix:

$$U = \begin{pmatrix} -1 & 1 & 0 & \cdots & 0 \\ -1 & 0 & 1 & \cdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ -1 & 0 & \cdots & 0 & 1 \end{pmatrix},$$

and $g_ju$ and $g_jX$ are the derivatives of the price-dividend ratio $g_j$ with respect to $u$ and $X$, respectively, which can also be obtained in semi-closed form. Time-varying disaster risk and disagreement endogenously generate time variation in the diffusion volatilities and jump sizes of stock returns, even if the parameters in the dividend growth processes are constant.

### 3 Results and Analysis

In this section I study the properties of the risk premia and other asset pricing implications of the model presented in Section 2. I analyze its main qualitative implications and the mechanisms behind them by means of a simple numerical illustration for a symmetric economy with two stocks, $N = 2$. In the baseline calibration, dividend growth processes have a drift $\mu = 2.5\%$ and a diffusion volatility $\sigma = 5\%$. Rare events have an impact of $k = -0.41$, consistently with the estimates reported in Dieckmann and Gallmeyer (2005) and in Barro (2006). Idiosyncratic jumps have a constant intensity $\lambda = 1\%$ and systematic jumps occur with a long-term frequency $\beta^A = 1\%$, so jumps in individual dividend processes occur on average each fifty years. The optimistic agent believes that

---

18. The values for the diffusion component of the dividend dynamics, $\mu$ and $\sigma$, are within the ranges considered in the literature. See, among others, Campbell (2003) and Cochrane, Longstaff, and Santa-Clara (2008).

19. This value is often criticised in the literature, but the results of this paper, at least qualitatively, are not affected by this debate. The patterns that I document could be obtained also assuming a lower jump size and a slightly higher jump intensity.
the long-term mean of the frequency of systematic disasters is smaller than does the pessimistic agent, i.e. \( \beta_B = 0.01\% \). The two agents have the same CRRA preferences along with a time horizon \( T \) of 50 years, a time preference rate \( \delta = 4\% \), and a risk aversion parameter \( \gamma = 4 \). The parameters of the \( X \) process are \( \varphi = 0.142 \) and \( \sigma_X = 0.05 \). Preference parameters are taken from Chen, Joslin, and Tran (2012), while the parameters in the \( X \) process are chosen to match the properties of Chen, Joslin, and Tran (2012)’s calibrated time-varying disaster intensity. Model parameters are summarized in Table 1.

### 3.1 Equity and variance risk premia

From agent \( i \)’s perspective, the risk premium for any security is defined as the difference between the expected return under \( \mathbb{P}^i \) and under the risk-neutral measure \( \mathbb{Q} \). I report risk premia relative to agent \( A \)’s beliefs, \( \mathbb{P}^A \). Define the cum-dividend instantaneous return of stock \( j \) as

\[
dR_{jt} = \frac{dS_j(t) + D_j(t)dt}{S_j(t)}.
\]

The instantaneous conditional equity risk premium of the individual stock \( j \), \( ERP_j \), is thus

\[
ERP_{jt} = E_t^A(dR_{jt}) - E_t^Q(dR_{jt}) = \gamma \sigma^2 \left( e'_j + \frac{g'_{ij}U}{g_j} \right) s - \lambda \sum_{i=1}^{N} k_{S_{j},i}(t)(JP_{it} - 1) - \lambda_c^A(t)k_{S_j}(t)(JP_{ct} - 1),
\]

where \( s = (s_1, s_2, \ldots, s_N)' \) is the vector of dividend shares, \( JP_{it} = \lambda_i^Q(t)/\lambda \) is the jump premium related to an idiosyncratic jump in the dividend growth of asset \( i \), and \( JP_{ct} = \lambda_c^Q(t)/\lambda_c^A(t) \) is the jump premium related to a common jump. The first term in (27) is the compensation for diffusion risk; the other terms represent a premium for bearing idiosyncratic and systematic disaster risk, respectively.

As mentioned in Section 2.4, the jump premium for idiosyncratic event risk, \( JP_{it} = (s_k + 1)^{-\gamma} \), is always greater than 1 and it is also close to 1 for small stocks. We can use Equation (21) to write the jump premium for systematic event risk as

\[
JP_{ct} = \left[ 1 + \left( \frac{\phi(t)^{\frac{\beta_A}{2\pi}}}{1 + \phi(t)^{1/\gamma}} \right)^{1/\gamma} \right]^{\gamma} (k + 1)^{-\gamma} = (k_{C,A}(t) + 1)^{-\gamma},
\]

(28)
where
\[ k_{CA}(t) = \frac{(1 + \phi(t)^{1/\gamma})(k + 1)}{1 + \left( \phi(t)^{\beta B / \beta A} \right)^{1/\gamma}} - 1 \]
is the size of the jump in equilibrium consumption of agent A in response to a systematic disaster. This jump size varies depending on the level of disagreement and the consumption share distribution, due to risk sharing between agents. Since agent B (the optimist) thinks systematic disasters are highly unlikely, he is willing to give up consumption in future systematic disaster states in exchange for higher consumption in all other future states. This mechanism reduces the consumption loss of agent A in the event of a systematic disaster and lowers the corresponding jump risk premium. The more wealth the optimist has, the more disaster insurance he is able to sell. So when the wealth share of the optimist is high, consumption of agent A can even increase at a disaster. That scenario would lead to a jump premium lower than 1—in other words, to a risk-neutral intensity \( \lambda_c^Q \) lower than the physical intensity \( \lambda_c^A \).20 A higher level of relative risk aversion \( \gamma \) would lead to a much faster rise in the systematic jump premium, although the qualitative implications would remain unchanged.

Figure 1 shows the conditional instantaneous equity premium of stock 1 at time \( t = 0 \) and its components, as a function of the dividend share \( s_1 \), for two possible values of the initial wealth share of the pessimistic agent A (\( c^A = 0.1 \) in the left panel and \( c^A = 0.9 \) in the right panel). The equity premium is first slightly decreasing and then increasing in the dividend share of the asset; a pattern that is due to the behavior of the compensation for diffusion risk (see Martin (2013)) and to the fact that the overall premium for idiosyncratic risk is lower for intermediate values of the dividend share. Note that the compensation for diffusion and idiosyncratic rare event risk does not change with the consumption share of the two agents, since they disagree only with respect to the systematic disaster intensity. The contribution to stock 1’s equity premium of its own idiosyncratic jump risk starts

20More precisely, the systematic jump premium is less than 1 when the ratio of the consumption shares is large:
\[ \frac{c^B(t)}{c^A(t)} > \frac{-k}{k + 1 - \left( \frac{\beta B}{\beta A} \right)^{1/\gamma}} \]
if the disagreement is large enough, that is, if \( \frac{\beta B}{\beta A} < (k + 1)^{\gamma} \). In the calibration this condition is satisfied when the consumption share of the optimist, \( c^B(t) \), is at least 60%. 

20
at zero but increases substantially with its dividend share, as the asset becomes more systematic. The compensation due to idiosyncratic rare event risk in asset 2’s dividends is small unless the second stock contributes to a large fraction of aggregate consumption. On the other hand, the component of asset 1’s equity premium that is due to systematic rare event risk is basically flat with the dividend share but depends on disagreement risk and reflects risk sharing between agents. The compensation for systematic jump risk is negative for small consumption shares of the pessimist but increases rapidly, and for large values of $c^A$ that compensation accounts for a large fraction of the individual equity premium (mainly when dividends are evenly distributed between the two stocks). This effect is primarily driven by the jump premium for systematic disasters, $JP_{ct}$. Besides the jump risk premia, the equity premium is also a function of the jump sizes of stock returns $k_{S_{jt}}$ and $k_{S_{jt}}^c$, which depend on the dividend loss and on changes in the price-dividend ratios, as shown in Equations (26) and (25). Under CRRA utility, the drop in the risk-free rate following a systematic disaster can dominate the effect of a rising risk premium, which would lead to a higher price-dividend ratio. This partially offsets the drop in dividends, making the return less sensitive to systematic disasters. The variation of systematic jump size in stock returns with the pessimist’s consumption share is stronger for a small stock and depends crucially on the assumption of difference in beliefs. In fact, if there is no disagreement then the systematic jump size $k_{S_{jt}}^c$ is constant and equal to the loss in dividend growth $k$.

In the same way, the instantaneous variance risk premium of stock $j$, $VRP_{jt}$, can be computed as the difference between objective and risk-neutral expectations of the return variance:

$$VRP_{jt} = E_t^A[(dR_{jt})^2] - E_t^Q[(dR_{jt})^2]$$

$$= -\lambda \sum_{i=1}^N k_{S_{jt},i}(t)^2(JP_{it} - 1) - \lambda^A_c(t)k_{S_{jt}}^c(t)^2(JP_{ct} - 1).$$

Given the assumption of constant dividend growth volatilities, the variance risk premium depends only on the jump risk components. Yet empirical evidence reported in Bollerslev and Todorov (2011) shows that compensation for rare events actually accounts for a large fraction of variance risk premia. The instantaneous variance premium $VRP_{jt}$ is usually negative, as expected, but it can become positive when $JP_{ct} < 1$ and large
enough to balance out the contribution of the idiosyncratic jump components, which is always negative. The variance risk premium is negatively related to the systematic jump premium: it decreases with agent A’s consumption share, and it is either decreasing or hump-shaped with respect to a stock’s dividend share (depending on the value of the calibrated parameters). As for the individual equity premium, the compensation due to idiosyncratic rare event risk in asset 2’s dividends is nearly zero; however, the contribution of idiosyncratic rare event risk in its own dividend process is increasing (in absolute value) in the dividend share (see Figure 2).

The model relates the correlation between individual variance premia to the systematic rare event risk. This systematic component is stronger when the consumption share of the pessimist is higher. In the case of a two-stocks economy, the average model-implied correlation between variance premia ranges from $-0.4$ (when the consumption share of agent A is 10%) to about 0.75 (when the pessimist consumes 90% of the aggregate dividend).

Now let me define the instantaneous return on the stock market index as the weighted sum of all individual asset returns:\(^{21}\)

$$dR_t = \sum_{j=1}^{N} s_j dR_{jt}. \quad (30)$$

The instantaneous equity premium on the index is then $ERP_t = \sum_{j=1}^{N} s_j ERP_{jt}$ and the instantaneous index variance risk premium is given by

$$VRP_t = \sum_{j=1}^{N} s_j^2 VRP_{jt} + \sum_{j=1}^{N} \sum_{i \neq j} s_j s_i CRP_{jit}, \quad (31)$$

where $CRP_{jit} = E_A[t_d R_{jt} dR_{it}] - E_Q[t_d R_{jt} dR_{it}]$ is the premium associated with the covariance between returns of assets $j$ and $i$.

Figure 3 plots the instantaneous equity (upper panels) and variance (lower panels) risk premium of the market, under agent A’s beliefs, as a function of the dividend share of asset 1, $s_1$, and their decomposition in terms of individual equity and variance premia for different values of the consumption share of the pessimistic agent. The market equity premium increases with the consumption share of the pessimist, and it is lower when

\(^{21}\)The stock market index can also be viewed as a claim on the aggregate endowment $C = D_1 + \cdots + D_N$. 22
the two assets contribute in the same way to the aggregate dividend because the equity premium of individual stocks grows more than linearly with the dividend share. The same reasoning holds for the absolute value of the aggregate variance premium—which includes, however, an additional component reflecting the priced covariance between stock returns. The covariance premium can contribute to a large portion of the aggregate variance premium when the economy is dominated by the pessimistic agent, mostly when the number of assets increases and they are relatively evenly distributed.

Apart from the aggregate variance premium’s dependence on the relative dividend and consumption shares, its dynamic properties are worth examining. I simulate 30-year paths of the variance risk premium at a monthly frequency from the model while using calibrated parameters for different values of the initial wealth share of the pessimistic agent, \( c_A \). Table 2 shows that, as the consumption share of the pessimist increases, the VRP is both larger (in absolute value) and more volatile. A systematic disaster induces an upward jump in the consumption share of the pessimist. That leads to a downward jump in the variance risk premium, which is then followed by more negative and volatile premia. Despite the setting’s simplicity, the dynamics of model-implied premia resembles the behavior of observed variance risk premia (see Section 4 and Figure 6), in which periods of low and smooth premia seem to be followed by larger and more volatile values. Empirically a regime switch often corresponds to the beginning of a crisis, so it could be linked to a systematic jump in the endowment process.

3.2 Stock return correlation and correlation risk premium

From Equation (15), the instantaneous conditional correlation between returns of stock \( i \) and stock \( j \) is given by

\[
\text{Corr}_i(t, dR_{it}, dR_{jt}) = \frac{\sigma_{S_i}(t, S_j(t)) + k_{S_i}(t)k_{S_j}(t)\lambda + k^2_{S_i}(t)\lambda^2}{\sqrt{\left(\sigma_{S_i}(t) + k_{S_i}(t)\lambda + k^2_{S_i}(t)\lambda^2\right)^2 + k_{S_j}(t)\lambda^2}}.
\] (32)

The first panel in Figure 4 shows the conditional stock return correlation in a symmetric economy with \( N = 2 \) stocks as a function of the first tree’s dividend share \( s_1 \) and the pessimistic investor’s consumption share \( c_A \) while using the model parameters in Table 1. A comparative statics analysis reveals that disagreement reduces stock return correlation
on average and in particular when risk sharing is stronger—that is, when the consumption
shares of the two agents are similar. Overall, the correlation under the pessimistic agent’s
objective measure is relatively flat: it has values between 35% and 43% and an average
across all states of about 39%.

The correlation risk premium is defined as the difference between the instantaneous
conditional correlation computed under the physical and the risk-neutral measure,

\[ \text{Corr}_t^{RP_{ij}}(dR_{it}, dR_{jt}) = \text{Corr}_t^{A}(dR_{it}, dR_{jt}) - \text{Corr}_t^{Q}(dR_{it}, dR_{jt}). \]  (33)

Here the risk-neutral correlation is computed as in (32) but using the risk-neutral idiosyn-
cratic and systematic rare event intensities \( \lambda_{ij}^{Q} \) and \( \lambda_{ij}^{Q_c} \), respectively (see second panel of
Figure 4). The model-implied risk-neutral correlation increases substantially with the
consumption share of the pessimistic agent and ranges approximately between 11% and
75%. This result is consistent with the empirical findings of Driessen, Maenhout, and
Vilkov (2012), who show that the implied correlation for the S&P500 is highly coun-
tercyclical and fluctuates between 0.2 and 0.8 for the period 1996–2010. The dynamics
of the risk-neutral correlation is almost entirely driven by disagreement between agents
about the probability of a systematic disaster. The average risk-neutral correlation for
the full model is about 46%, which corresponds to an average instantaneous correlation
risk premium of about –7%; this value, too, is consistent with the empirical findings
reported by Driessen, Maenhout, and Vilkov (2012). However, the model-implied corre-
lation premium (see third panel of Figure 4) can be much larger in absolute value when
the pessimist accounts for a large part of the aggregate consumption, and it can also
become positive when the pessimist’s consumption share is relatively low—mainly when
the dividend shares of the two assets are similar.

3.3 Relation between the equity and the variance risk premium

Comparing the expressions for the variance premium (see Equations (29) and (31)) with
those for the equity premium (Equation (27)) shows that rare event risk implies a tight
link between the two, both for the market and for the cross section of stock returns. This
link provides our basic intuition for the role of the variance premium in predicting future
excess returns, which is consistent with Bollerslev, Tauchen, and Zhou (2009)’s empirical
finding that aggregate variance risk premium can explain a nontrivial fraction of the
time-series variation in post-1990 aggregate stock market returns. Premia that are high
(in absolute value) predict high future returns—though mainly over short horizons, when
the compensation for rare events accounts for a large portion of the empirical equity and
variance risk premia (see e.g. Bollerslev and Todorov (2011)). Yet standard predictive
regressions imply an unconditionally linear relation between equity and variance risk
premia, whereas the model’s relation is conditional on the information set at time $t$.
The idiosyncratic and systematic event risk components of the equity and variance risk
premium are linearly related, but the regression coefficients are stochastic and given
by the inverse of the corresponding jump size. Depending on which of the jump risk
components dominates, the relation can be either weaker or stronger. The importance
of the idiosyncratic and systematic rare event risk contribution to the risk premia, as
well as the jump sizes in stock returns, are both functions of the asset’s dividend share
and the agents’ consumption share. Thus, time variation in share distributions leads to
a time-varying relation between equity and variance risk premia, both at an aggregate
level and for individual stocks. For individual stocks, however, empirical estimates of the
variance premium are noisy owing to lack of reliable high-frequency data for computing
the realized variance. Moreover, there is evidence of a large systematic component in the
cross section of variance risk premia (see e.g. Carr and Wu (2009)). Hence this paper
explores the relation between the instantaneous equity premium of individual stocks and
the market’s variance risk premium.

To develop a better understanding of the model implications that concern the predic-
tive power of aggregate variance premium for market and individual stock excess returns,
I run regressions on simulated data. This involves simulating 30 years of monthly excess
stock and market returns and the instantaneous variance risk premium from the model
in Section 2 while assuming a symmetric economy with $N = 2$ stocks and using the base-
line model parameters. The purpose of these simulations is to investigate the model’s
qualitative implications for the interaction between the aggregate variance premium and
the excess stock and market returns. This is a natural step between the model and the
empirical evidence presented in Section 4. I consider return regressions of the form

$$r_{i,t+h}^e = \alpha_i + \beta_i \ VRP_t + \varepsilon_{i,t+h},$$

25
where \( r_{i,t+h} \) is the simulated log excess return of the two stocks \((i = 1, 2)\) or of the market \((i = M)\) and where \( h \) is the return horizon in months. The excess return is given as an annualized percentage and the variance premium is given as a monthly squared percentage for consistency with the literature (see e.g. Bollerslev, Tauchen, and Zhou (2009) and Drechsler (2013)). Table 3 reports the average regression coefficient and adjusted \( R^2 \) (with standard errors in parentheses) for the market at horizons \( h = 1, 6, \) and 12 months and for different values of the initial consumption share of the pessimistic agent, \( c^A = 0.1, 0.5 \) and 0.9. The predictive coefficient is generally negative. The predictive power increases with the horizon and with the initial wealth share of the pessimistic agent, which is also associated with larger (absolute) values of the variance risk premium and of its volatility (again, see Table 2), and at 6 and 12 months horizon the regression coefficient is significantly different from zero only when the pessimist holds a large fraction of the aggregate endowment. Note that the average estimated regression coefficient is quite close to what is found in the data (see Section 4), even if the model is not estimated or calibrated to match the observed \( VRP \) moments.

Turning now to the cross section, the first two panels of Table 4 display results of the same predictive regressions for the two individual stocks in the economy. Initially, the small stock (Panel A) has dividend share \( s_1 = 0.1 \) and the big stock (Panel B) has a share \( s_2 = 0.9 \). The regression coefficient for the small stock is often positive and not significant. The reason is that, with only two stocks in the economy, the fear of idiosyncratic disasters in the dividend growth of the large stock has a strong effect on the equity premium of the small stock; the corresponding jump size in the small stock return, \( k_{S_1,2}(t) \), can be positive and thereby lead to a weak positive relation between the equity premium of the small stock and the market variance risk premium. This effect holds also after eliminating idiosyncratic jump risk \((\lambda = 0)\) for small values of the consumption share of the pessimist, because in that case the variance risk premium of the small stock is negatively correlated with the market variance premium. In contrast, the predictive regression results for the large stock (Panel B) are in line with those discussed for the market return regression, since the big stock contributes to a large fraction of the aggregate dividend.

An economy consisting only of two stocks, one of which accounts for 90% of aggregate consumption, is clearly not realistic. It would be interesting to run cross-sectional
predictive regressions for an economy with many assets and relatively small values of the dividend share, since these features better characterize real-world markets. However, it is not computationally feasible to simulate the model for large $N$ because the solution would require numerical evaluation of a high-dimensional integral at each time step. However, Section 3.4 investigates theoretically the special case of a large and diversified economy and demonstrates that, as $N$ increases, the idiosyncratic jump premium contribution the both equity and variance risk premia vanishes and the aggregate variance risk premium is almost entirely driven by a covariance premium. So in order to mimic the case of a large economy without the need to simulate it, I look at the predictive power of the simulated covariance risk premium for the excess return of individual stocks. Panels C and D of Table 4 display results of the regression

$$r^e_{i,t+h} = \alpha_i + \beta_i CRP_t + \epsilon_{t+h},$$

where $r^e_{i,t+h}$ is the simulated log excess return of the small and the big stock and $CRP$ is the covariance risk premium (in monthly squared percentage). For the small asset (Panel C), the predictive coefficient is negative and significantly different from zero. Its average value is similar across horizons and consumption shares of the pessimist, but the standard deviation of the regression coefficient decreases with $c^A$ leading to high adjusted $R^2$ when the pessimist accounts for a large share of aggregate consumption. On average, the $R^2$ values are even higher than those reported in Table 3 for the aggregate market. At the 6-month horizon, for example, the average adjusted $R^2$ for the regression of excess small asset returns on the instantaneous covariance premium is almost 19%, as compared with a 14% $R^2$ for the regression of market excess returns on the variance premium. For the large asset (Panel D), results are much weaker. Regression coefficients are even positive for small values of the pessimist’s consumption share yet become negative (but only marginally significant) for large $c^A$. These results indicate that, in a relatively large economy, the forecasting power of the aggregate variance risk premium for future excess returns should be stronger for small stocks because their returns are more dependent on the compensation for systematic rare event risk (though mainly for large values of the consumption share of the pessimistic agent).
3.4 The case of a large economy: \( N \to \infty \)

Let me now consider analytically the case in which the number of assets in the economy approaches infinity and dividends are evenly distributed across assets; that is, \( s_j = 1/N \) for \( j = 1, \ldots, N \). In this case, the premium for idiosyncratic risk in individual assets vanishes because \( JP_{jt} = (k/N + 1)^{-\gamma} \) converges to unity for all \( j \). Moreover, the diffusion component in the equity premium for stock \( j \) reduces to \( \gamma \sigma^2 / N \), which tends to zero as the number of assets \( N \) increases. Hence the expressions for the equity risk premium of stock \( j \) and of the index can be simplified as follows:

\[
\begin{align*}
ERP_{jt} &= -\lambda c(t) k_{S_j}(t)(JP_{ct} - 1), \\
ERP_t &= \gamma \sigma^2 - \lambda c(t) k_{S_j}(t)(JP_{ct} - 1).
\end{align*}
\]

(34)

(35)

The equity premium is the same for any stock because in this special case, \( k_{S_j}(t) = k_{S_i}(t) \) for all \( i \) and \( j \). Furthermore, the market equity premium is equal to the equity premium of single stocks plus the standard (constant) compensation for diffusion risk, \( \gamma \sigma^2 \), that arises in economies where dividend growth follows a geometric Brownian motion. Note that even if stocks are negligibly small, they still earn a risk premium due to the presence of the systematic jump component, which does not depend on the dividend share (see the black dashed-dotted lines in Figure 1).

Similarly, the variance risk premium is the same for any individual stock,

\[
VRP_{jt} = -\lambda c(t) k_{S_j}(t)^2 (JP_{ct} - 1),
\]

(36)

and is equal to the premium for the covariance between stock \( i \) and stock \( j \), \( CRP_{jit} = VRP_{jt} \). Then the variance premium for the market index becomes

\[
\begin{align*}
VRP_t &= \sum_{j=1}^{N} s_j^2 VRP_{jt} + \sum_{j=1}^{N} \sum_{i \neq j} s_j s_i CRP_{jit} \\
&= \frac{VRP_{jt}}{N} + \frac{N - 1}{N} CRP_{jit} \\
&= -\lambda c(t) k_{S_j}(t)^2 (JP_{ct} - 1),
\end{align*}
\]

(37)

which is equal to the variance risk premium of any individual stock and also to the covariance premium. In particular, from the second line of Equation (37) it is evident that, as \( N \) increases, all the market variance risk premium is due to a premium for
covariance. In accordance with this model-implied feature, Driessen, Maenhout, and Vilkov (2012) show empirically that the variance risk premium for the S&P500 index can be largely attributed to the high price of correlation risk.

To clarify premia behavior in this special case as a function of the consumption share distribution, Figure 5 shows the equity premium of stock $j$ and of the index, the systematic jump premium, and the index variance risk premium as functions of the consumption share of the pessimistic agent, $c^A$. In this case the link between variance risk premia and excess stock returns, both for the index and for single stocks, is straightforward:

\begin{align}
ERP_t &= \gamma \sigma^2 + \frac{1}{k_x(t)} VRP_t, \quad (38) \\
ERP_{jt} &= \frac{1}{k_x(t)} VRP_t. \quad (39)
\end{align}

and it is linear conditionally on the information set at time $t$. In particular, the regression coefficient $1/k_x(t)$ depends only on the consumption share of the two agents. This relation is negative and stronger for large values of the consumption share of the pessimist; the maximum is around $c^A = 0.7$, above which the relation becomes weaker for extreme values of $c^A$.

For a large and diversified index such as the S&P500, the model thus suggests a stronger predictive power of variance risk premium for future excess returns in periods during which pessimists have a relatively large consumption share—that is, in bad states of the economy, which are also generally linked to higher (absolute) values of the variance risk premium. I investigate this intuition empirically in Section 4.2.

### 3.5 Consumption share dynamics and survival

Agent survival is an important issue in complete markets models with heterogeneous beliefs and time-separable preferences (see e.g. Yan (2008) and Kogan, Ross, Wang, and Westerfield (2006)). Under most models in which agents have identical CRRA preferences, only those agents whose beliefs are closest to the truth will survive in the long

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22Numerical results are obtained for the parameters in Table 1 and $N = 10$, which is not that large but already entails solving a 9-dimensional integral—even though in the special case of equal dividend shares, the expression for the price-dividend ratio is simpler than in the general case in Section 2 (see Supplemental Appendix). Nonetheless, already for $N = 8$ or 9 the results are nearly identical.
run. If the irrational agent (optimist in the foregoing analysis) is quickly eliminated from the economy then the price effects generated by trading between agents disappear. It is therefore worth analyzing the survival of agents A and B, which is defined as their asymptotic share of consumption as the horizon goes to infinity (see e.g. Berrada (2009) and Dumas, Kurshev, and Uppal (2009)). Table 5 shows the mean and standard deviation of the share of consumption of the optimistic agent, $c^B$, at horizon $T = 50, 100, \text{ and } 500$ years; these values are obtained from 1,000 simulations starting from $c^B = 0.1, 0.5, \text{ and } 0.9$. I find that the optimist can survive for long periods and that his consumption share actually increases if there are no systematic disasters. Therefore, the risk-sharing dynamics documented previously are not likely to disappear quickly.

The dynamics of the consumption share of the pessimistic agent is obtained by applying Itô’s lemma to $c^A = (1 + \phi(t)^{1/\gamma})^{-1}$ and using Equation (4):

$$dc^A(t) = -\frac{1}{\gamma} c^A(t)c^B(t)(\beta^A - \beta^B)X(t)dt + c^A(t)c^B(t)\left(1 - \left(\frac{\beta^A}{\beta^B}\right)^{1/\gamma}\right)dN_t. \quad (40)$$

The drift is negative; thus $c^A$ declines deterministically when there is no systematic disaster but increases in response to systematic disaster, and both effects are stronger as the level of disagreement increases.

However, the consumption share’s distribution is not stationary, which means that at infinite horizon one of the agents eventually disappears. Such nonstationarity could potentially be an issue in light of an estimation of the model. Possible solutions are provided by Borovicka (2012), who shows that recursive preference specifications lead to equilibria in which both agents survive, and by Garleanu and Panageas (2015), who propose an overlapping-generations framework to obtain a nondegenerate stationary equilibrium. I leave extensions of the model in these directions to future research.

### 4 Empirical Analysis

This section briefly introduces the data before studying the empirical support of the main model’s implications. In particular, I first analyze the link between equity and variance risk premia at an aggregate level via predictive regressions of market excess returns on the
variance risk premium. Second, I investigate cross-sectional variations in the forecasting power of the aggregate variance premium.

4.1 Data

The empirical analysis is based on the aggregate $S&P500$ composite index (a proxy for the aggregate market portfolio) and on CRSP cap-based portfolios returns, to analyze the differential effects of small versus big stocks. I use monthly data from January 1990 through December 2011 for a total of 264 monthly observations. Excess returns are constructed by subtracting the log 30-day T-bill yield to the monthly returns, all obtained from CRSP.

The variance risk premium for any asset is defined (see also Section 3.1) as the difference between physical and risk-neutral expectations of total return variance for a given horizon. The Volatility Index (VIX), from the Chicago Board Options Exchange (CBOE) provides a model-free measure of the risk-neutral expectation of total market return variation over the subsequent 30 days and is based on the highly liquid $S&P500$ index options.\footnote{The VIX is subject to some approximation error (see, e.g., the discussion in Jiang and Tian (2007)), but relying on the squared VIX as a measure of the risk-neutral expected variance facilitates comparison with other studies. The VIX is reported in terms of annualized percentage volatility, therefore I square it and then divide by 12 to obtain a monthly quantity.} A measure of the realized variance of the market for a given month can be obtained by summing up $S&P500$ squared five-minute log returns\footnote{I obtain a monthly time series of realized variance based on five-minute returns from Hao Zhou’s webpage: \url{https://sites.google.com/site/haozhouspersonalhomepage/}.} and I compute the expectations under the physical measure of total stock market return variance by a simple projection of the realized variance measure on a set of predictor variables. As in Drechsler (2013) and Drechsler and Yaron (2011), the realized variance is projected on the value of the squared VIX at the end of the previous month and on a lagged realized variance measure.\footnote{I also implement the same regressions using an expanding window to rule out any look-ahead bias. Because the results are almost identical, I use the in-sample estimates to facilitate comparison with existing studies and to avoid losing observations at the beginning of the sample for the initial estimation.} The difference between the conditional forecast from the projections and the risk-neutral expectation, measured using the VIX, yields the series of one-month
market variance premium estimates plotted in Figure 6.\footnote{Similar dynamics are obtained when using more sophisticated models for the realized variance forecasts, such as the heterogeneous autoregressive (HAR) model of Corsi (2009), but here I focus on the simplest measure because more complex models are difficult to identify using monthly data. Bekaert and Hoerova (2014) compare different volatility forecasting models and show that the projection I use is the best within the simple specifications and also performs relatively well in comparison with more sophisticated models.} The variance premium for the market is negative on average, which means that investors are willing to pay a premium to be insured against high-variance states; the premium is time varying, with periods of a small and smooth premium alternating with periods in which the variance premium is larger (in absolute value) and more volatile. These phases of high and volatile variance premia seem to coincide with periods of large disagreement between investors, denoted by the light gray shaded areas in Figure 6. Periods in which differences in beliefs are large include recessions (denoted by dark gray shaded areas) and other times of financial distress, such as the Long-Term Capital Management crisis and Russian default in 1998.

I study empirically the forecasting power of the market variance risk premium for both the market excess return and cap-based portfolio returns.

Since the model-implied variance risk premium includes compensation only for jump risk, as a robustness check I run the same predictive regressions using the time series of market variance risk premium due to large jumps as computed by Bollerslev and Todorov (2011), although data are available only for the period 1996–2007.\footnote{I thank Viktor Todorov for providing the data.} Details on the data and a summary of the results using this alternative measure of the variance risk premium are provided in the Supplemental Appendix and are generally consistent with the results discussed in this section.

Proxies of belief disagreement are calculated using the mean absolute deviation of one-year-ahead forecasts on real GDP growth from the BlueChip Economic Indicator, which are available at a monthly frequency through December 2009.\footnote{See Buraschi and Whelan (2011) for details on the database, disagreement measures, seasonal adjustment, and construction of forecasts at fixed one-year horizons. I am grateful to Andrea Vedolin and Paul Whelan for providing the time series of belief disagreement on GDP growth.} Being consistent with the model presented in Section 2 would normally require that I measure disagreement about the perceived probability of a systematic disaster, but this is proportional to disagreement...
about the total expected consumption growth provided agents do agree on the expected
growth rate in normal times (see Equation (8)).

4.2 Predictive regressions for the market

The simple general equilibrium model in Section 2 implies a tight link between variance
risk premia and excess returns of the stock market index, which is consistent with the
empirical findings in Bollerslev, Tauchen, and Zhou (2009), Drechsler (2013), and Drech-
sler and Yaron (2011). Panel A of Table 6 displays results from ordinary least-squares
(OLS) estimation of standard return predictability regressions of the form

$$r_{t+h}^e = \alpha + \beta VRP_t + \epsilon_{t+h}. \quad (41)$$

I regress monthly S&P500 excess returns—at horizons $h$ ranging from one month to one
year—on the variance risk premium. The excess return series for $h > 1$ are overlapping, $t$-
statistics are Newey–West corrected, and I report adjusted $R^2$ in percentage. In line with
results reported in the literature, there is a negative and significant relation between
the variance premium and excess returns; also, the predictive power (measured either
as the adjusted $R^2$ or as $t$-statistics of the regression coefficient) is highest at the six-
month horizon.\(^{29}\) Panel B reports results from robust regressions that employ Huber-type
weights to limit the influence of outliers. The robust regression estimates agree both in
magnitude and sign with the OLS estimates, and in most cases the predictability evidence
is even stronger.\(^{30}\) Overall, these results indicate a considerable ability of the variance
risk premium to predict future market excess returns.

Estimating the regression in Equation (41) however implicitly imposes major restric-
tions on the relation between variance risk premium and future returns, since that re-
gression assumes a monotone and linear structure. The theoretical asset pricing model

\(^{29}\)Considering only the pre-crisis sample the $R^2$ peaks at a quarterly horizon and the regression
coefficients become insignificant at long horizons; results are available upon request and are consistent
with those of Bollerslev, Tauchen, and Zhou (2009), whose sample ends in December 2007.

\(^{30}\)A more naive way to control for the effect of outliers is to run the OLS regression without the two
potentially anomalous observations of October and November 2008 (at the peak of the financial crisis)
when the realized variance experienced unprecedented levels. Results do not qualitatively change, and
the estimates are just slightly more significant than in Panel A of Table 6.
presented in Section 2 suggests that this relation is only conditionally linear; unconditionally the relation need not be linear or even monotonic. Therefore, I study empirically potential instabilities or nonlinearities in the standard regression results. Introducing additional regressors does not qualitatively change the results (as shown by Bollerslev, Tauchen, and Zhou (2009)), so I rely on simple regressions to outline the properties of the relation between returns and variance premia. I also focus on the six-month horizon, for which significance of the standard predictive regressions seems to be stronger. The regression coefficient $\beta$ for a large market index should vary with the distribution of the consumption share between agents (see Section 3.4). That distribution is not observable, but in the model it is directly linked to the level and volatility of the variance risk premium (see Table 2); hence I investigate the shape of the predictive relation for different levels of the premia.

First, I run regression (41) separately for different quantiles of the variance risk premium. Figure 7 plots the distributions of regression coefficients (upper panel) and $R^2$ (lower panel), which are obtained by applying a block bootstrap procedure. In both panels, the leftmost box plot corresponds to small absolute values of the premium ($VRP < q_{70\%}$), the rightmost one to large values ($VRP > q_{30\%}$), and the middle box plot to average values of the $VRP$. In accordance with the model, predictive power is increasing in the (absolute) level of the variance premium and the regression coefficient is significantly different from zero only for large values of the variance risk premium. From an empirical standpoint, changes in the variance premium are of course not exclusively related to changes in the cross-sectional consumption distribution of disagreeing agents. In order to relate more tightly the instability of standard predictive regressions to the extent of risk sharing among agents, I stratify regression (41) according to the level of difference in beliefs (DB). Again, the regression coefficient increases in absolute value with the level of DB, from $-0.2415$ to $-0.9725$; the adjusted $R^2$ is 1.09% for small DB and increases to 4.02% and 20.91% for average and large DB, respectively.\(^{31}\) The link between the level of the variance premium and measures of disagreement is confirmed by a simple OLS regression of monthly $VRP$ on DB from January 1990 through December

\(^{31}\)The values of the regression coefficients and $R^2$ are not exactly comparable to the results obtained previously because DB is available only until December 2009 (see Section 4.1).
2009. A change of one standard deviation in DB yields a change of 0.38 standard deviations in the variance premium; this result is strongly significant both statistically and economically, with a Newey–West corrected $t$-statistic of $-4.319$ and an adjusted $R^2$ of about 14%.

A second way to analyze the validity of a simple linear regression is to estimate a fully nonparametric regression of the form

$$r^e_{t+h} = m(VRP_t) + \varepsilon_{t+h},$$

where $m: \mathbb{R} \rightarrow \mathbb{R}$ is an arbitrary function fulfilling some smoothness conditions. An estimate of the function $m$ can be simply obtained by using the Nadaraya–Watson kernel estimator with, in this univariate case, a Gaussian kernel; see Figure 8. The number of observations is not large enough to draw strong conclusions, but a visual inspection clearly confirms the absence of any link between the two variables for small (absolute) values of the variance risk premia, although there is a stronger negative relation for more extreme values of those premia. Hence this simple nonparametric analysis supports the conclusion that the predictive power of variance risk premia for market returns is a time-varying and nonlinear phenomenon.

To avoid a fully nonparametric procedure, it is possible to model explicitly the regression coefficient’s time variation. The conditional $\beta$ could, for example, be a function of the variance premium itself:

$$\tilde{r}^e_{t+h} = \left(\beta_0 + \beta_1 \tilde{VRP}_t\right) \tilde{VRP}_t + \varepsilon_{t+h},$$

where the tilde marks variables that are standardized. Equation (43) is equivalent to a quadratic regression and can be estimated via standard OLS. If $\beta_1$ proved to be insignificant then we could not reject a linear relation between excess returns and lagged variance premia, but $\beta_1$ is actually both positive and significant whereas $\beta_0$ (the linear term) is insignificant; the adjusted $R^2$ of this regression is 9.47%, which corresponds to

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32The optimal bandwidth, computed as suggested by Bowman and Azzalini (1997), is 4.77.

33In other words, I am estimating Equation (42) while requiring that $m$ be a quadratic function. One could, theoretically, employ other functional forms, but this is the most obvious alternative to a linear regression.
a 22% increase over the linear regression’s adjusted $R^2$ of 7.75% (at six-month horizon). In general, time variation in the regression coefficient is modeled by introducing an interaction term. Apart from the level of $VRP$, other reasonable candidates worth exploring are the volatility of $VRP$ and the level of disagreement or optimism. If we use DB as a conditioning variable, then the regression coefficient $\beta_t$ becomes more negative with increasing disagreement (as expected), and the adjusted $R^2$ for the monthly 1990–2009 sample increases from 6.3% to 7.6%.

An alternative way of analyzing the time variation in the predictive power of the variance premium for future returns is to estimate a standard regression on a rolling window. I regress market excess returns at a 6-month horizon on lagged variance risk premium using a rolling window of 50 months. Figure 9 reports estimates of the slope coefficient and corresponding adjusted $R^2$. Instability of the predictive relation is evident, and it is possible to relate the time variation in the slope coefficient to measures of disagreement. The correlation between the rolling regression coefficient estimate and a moving average of the difference in belief measure is equal to $-51.18\%$, which means that the regression coefficient becomes more negative as disagreement increases. The relation between the regression’s slope coefficient and the difference in beliefs suggests that the predictive power of variance risk premia for future excess returns is countercyclical, given that measures of disagreement are known to increase in bad times (see e.g. Patton and Timmermann (2010) and Buraschi, Trojani, and Vedolin (2014a)).

A growing body of empirical evidence documents instabilities and nonlinearities in the strength of the return predictability by popular macroeconomic variables such as the dividend yield and short rate variables. For example, Henkel, Martin, and Nardari (2011) use a regime-switching model to show that standard aggregate return predictors are effective during business cycle contractions but practically useless during expansions. In the same way, I examine the dynamics in the predictive power of variance risk premia via estimation of a regime-switching model:

$$\tilde{r}_{t+h} = \beta_s \tilde{VRP}_t + \epsilon_{s,t+1},$$

(44)

where $\epsilon_s \sim N(0, \sigma^2_s)$ and the state $s \in \{1, 2\}$ follows a Markov chain with constant transition probabilities. I find that predictability is present only in state 2, which is char-
characterized by more volatile and larger (on average, in absolute value) variance risk premia. In state 1, the regression coefficient is positive and insignificant. State 2 corresponds to periods of financial crisis (the US savings and loans crisis in the early 90s; the Asian financial crisis, Russian default, and the bursting of the dot-com bubble from 1996 to 2002, and the recent financial crisis from late 2007); therefore, the shift in regime of the variance premium (and of its predictive power) could be linked to a systematic disaster or to a jump in the consumption share of pessimistic agents, as would be implied by the simple model in Section 2.

4.3 Predictive regressions in the cross section

I next study the predictive power of market variance risk premia for cap-based portfolio excess returns, estimating regressions of the form

$$r_{i,t+h}^e = \alpha_i + \beta_i VRP_t + \varepsilon_{t+h},$$

(45)

where $r_{i,t+h}^e$ denotes the monthly excess returns on portfolio $i$ with horizon $h$. Table 7 reports results for portfolios including deciles 1 and 2 (large-cap CRSP index), 3 to 5 (mid-cap CRSP index), and 6 to 10 (small-cap CRSP index). At the one-month horizon, there does not seem to be a clear pattern and overall significance is quite weak. At longer horizons, the predictive power of the market variance risk premium for excess returns is much stronger for small stocks, in line with the model and with the empirical evidence described previously. At the six-month horizon, for instance, adjusted $R^2$ for the small-cap portfolio is 10.17%—more than 50% larger than the 6.21% $R^2$ for the big-cap portfolio. The forecasting power with respect to small stocks is still impressive at the one-year horizon, with an $R^2$ of 9.67%. As a robustness check, I estimate the same regression on the 25 Fama and French portfolios that are sorted by the firm characteristics of size and book-to-market ratio (BM).\textsuperscript{34} Figure 10 plots estimates of $VRP$ loadings (left panel) and of adjusted $R^2$ in percentage (right panel) from regression (45), at the six-month

\textsuperscript{34}Data are obtained from Kenneth French’s website. The portfolios are the intersections of five portfolios formed on size and five portfolios formed on the ratio of book equity to market equity. Breakpoints are the NYSE market equity and $BM$ quintiles. Size 1 corresponds to small stocks and $BM$ 1 to growth stocks.
horizon, for the 25 Fama and French portfolios. Lines connect portfolios of different book-to-market categories within each size category while focusing on the bottom and upper quintiles, which correspond to small and big stocks, respectively. On average, small stocks have larger (in absolute value) VRP loading and higher $R^2$. This means that exposure to aggregate variance risk could partially explain the size premium.\(^{35}\) The predictive power of variance risk premium for future returns seems to be stronger also for growth stocks. Within the simulated model, however, no distinction can be drawn between size and value effects if assets have identically distributed cash flows.

As in the case of predictive regressions for the aggregate market, discussed in Section 4.2, the regression model in Equation (45) likewise assumes a monotone and linear structure. Therefore, also in the cross section I analyze the dependence of the linear regression coefficient $\beta_i$ on the level of the variance risk premium and of the difference in beliefs (focusing on the six-month horizon); results are consistent with those reported in Section 4.2. In accordance with the model and just as for the aggregate predictive regression, predictive power increases with the level of the difference in beliefs (see Table 8). Differences among the DB quantiles seem to be stronger for the small-cap portfolio. Also, the difference between small- and big-cap beta is significant only in the state where there is a large difference in beliefs. In other states, the VRP loading and the adjusted $R^2$ are similar for large and small stocks, and the big-cap portfolio beta is even higher (in absolute value) than the beta for small stocks when disagreement is low. The last panel of Table 8 reports results of regressing, on the aggregate variance risk premium, the return of a portfolio that is long the small-cap index and short the big-cap index. The regression coefficient is negative and strongly significant when disagreement is high, with an $R^2$ of more than 20%. Therefore, in these states a large (absolute) variance risk premium predicts a larger size premium, while the effect is not significant when difference in beliefs is low. This finding is related to the work of Lemmon and Portniaguina (2006), who show a negative relation between the size premium and consumer confidence. In fact, the size effect (of smaller firms having higher returns on average) seems to be concentrated in periods characterized by large disagreement, as shown in Table 9. It is natural for investors

\(^{35}\)Bali and Zhou (2015) show that an asset pricing model in which both market risk and aggregate variance risk premium are priced can explain the premia for industry, size, and value.
to require higher returns on assets that are more sensitive to systematic disaster risk. The economic intuition for this finding can be found in Jagannathan and Wang (1996), who argue that firms more likely to exhibit financial distress (e.g., small firms) have market betas that are more sensitive to changes in the business cycle. Investor sentiment, or disagreement, is thus related to time variation in the expected returns of those firms because these factors forecast future business conditions. However, this reasoning holds only when the perceived systematic jump premium is high. The model suggests that the systematic jump premium component could actually have a negative effect on the stock’s excess returns if the consumption share of pessimists were low enough. Thus the size premium could go in opposite directions depending on which agent type dominates the market. This finding is consistent with some of the later empirical research on the size effect, which suggests that the premium disappears in the 1980s.

5 Conclusion

This paper studies both theoretically and empirically how agent disagreement about the likelihood of systematic disasters affects the equity and variance risk premia and the relation between them, both for the market portfolio and in the cross section of stocks. The starting point is a general equilibrium model with multiple trees and disagreement about systematic rare event risk.

The main findings are the following. First, the equity (variance) risk premium of an individual stock tends to increase (decrease) with its dividend share and with the consumption share of the pessimistic agent. The variance risk premium can also switch sign, mainly for small stocks, and it is time varying; it alternates phases of small and smooth premia with periods in which the variance premium is larger (in absolute value) and more volatile. Second, the index variance risk premium is largely due to a covariance premium when assets are relatively evenly distributed or the number of stocks in the economy is large. The model-implied correlation risk premium, as the variance risk premium, is large (in absolute value) when pessimists hold a large fraction of the aggregate endowment. Third, rare event risk implies a tight link between the equity and the variance risk premia, both for the market and for the cross section of stock returns. This link
however is state-dependent and varies with the asset’s dividend share and the agents’ consumption share. In particular, the relation is stronger when the consumption share of the pessimist is larger, i.e., in bad states of the economy, and for small stocks. Fourth, in the case of a large diversified economy only systematic risk is priced and the relation between equity and variance risk premia is conditionally linear. Moreover, infinitely small assets still earn a risk premium owing to the presence of systematic rare event risk.

I investigate empirically the main predictions and show that, as implied by the model, the relation between the equity premium and the index variance premium is time varying and systematically linked to the degree of risk sharing among disagreeing investors. In particular, the predictive power of variance premium for future excess returns is stronger in periods of large differences in investor beliefs. This relation holds especially for small stocks, whose returns are more dependent on the compensation for systematic rare event risk.

This work suggests several interesting lines of future research. On the theoretical side, the model’s simplicity means that several extensions are possible. Examples include introducing a stochastic diffusion volatility of the dividend processes and allowing for learning based on an exogenous signal about the state of the economy. I could also introduce disagreement with regard to both the disaster intensity and the expected dividend growth in normal times. Given the link between volatility risk premia and option surfaces, the option pricing implications of an heterogeneous rare disaster model are also worth exploring.

On the empirical side, it would be natural to look for potential time variation and nonlinearity in the relation between stock returns and correlation risk premia as I find for the variance risk premium. It would also be worth investigating whether the same nonlinear relationships are present in other markets for which a link between disagreement or variance premia and excess returns has been documented, as for example the fixed income market (Mueller, Vedolin, and Yen (2011) and Buraschi and Whelan (2011)) and the foreign exchange market (Beber, Breedon, and Buraschi (2010)).
References


A Tables and Figures

Table 1: Model parameters

<table>
<thead>
<tr>
<th>Preferences</th>
<th>$\delta = 0.04$</th>
<th>$\gamma = 4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dividends</td>
<td>$\mu = 2.5%$</td>
<td>$\sigma = 5%$</td>
</tr>
<tr>
<td>Intensities</td>
<td>$\beta^A = 1%$</td>
<td>$\beta^B = 0.01%$</td>
</tr>
<tr>
<td></td>
<td>$\varphi = 0.142$</td>
<td>$\sigma_X = 0.05$</td>
</tr>
</tbody>
</table>

Table 2: Simulated market variance risk premium

<table>
<thead>
<tr>
<th>$\hat{c}^A$</th>
<th>0.1</th>
<th>0.3</th>
<th>0.5</th>
<th>0.7</th>
<th>0.9</th>
</tr>
</thead>
<tbody>
<tr>
<td>mean $VRP$</td>
<td>$-4.70$</td>
<td>$-5.48$</td>
<td>$-6.76$</td>
<td>$-8.85$</td>
<td>$-12.74$</td>
</tr>
<tr>
<td></td>
<td>(1.13)</td>
<td>(1.33)</td>
<td>(1.50)</td>
<td>(1.52)</td>
<td>(1.33)</td>
</tr>
<tr>
<td>std $VRP$</td>
<td>0.55</td>
<td>0.66</td>
<td>0.82</td>
<td>0.99</td>
<td>0.95</td>
</tr>
<tr>
<td></td>
<td>(0.27)</td>
<td>(0.47)</td>
<td>(0.65)</td>
<td>(0.71)</td>
<td>(0.42)</td>
</tr>
</tbody>
</table>
Table 3: Market return predictability by variance risk premium (VRP), from simulated monthly data, at horizons $h = 1, 6$ and 12 months, for different values of the initial consumption share of the pessimistic agent, $c^A = 0.1, 0.5$, and 0.9. The table shows the average of the regression coefficient and adjusted $R^2$ over all simulations, with standard errors in parenthesis. Returns are in annualized percentage while VRP is in monthly squared percentage.

<table>
<thead>
<tr>
<th>Horizon (months)</th>
<th>$c^A = 0.1$</th>
<th>$c^A = 0.5$</th>
<th>$c^A = 0.9$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$VRP$ Coeff</td>
<td>-0.51</td>
<td>-0.38</td>
<td>-0.35</td>
</tr>
<tr>
<td></td>
<td>(0.19)</td>
<td>(0.12)</td>
<td>(0.08)</td>
</tr>
<tr>
<td>Adj $R^2$ (%)</td>
<td>2.15</td>
<td>3.15</td>
<td>4.26</td>
</tr>
<tr>
<td></td>
<td>(0.98)</td>
<td>(1.54)</td>
<td>(1.80)</td>
</tr>
</tbody>
</table>
**Table 4: Predictability of excess returns of small (Panels A and C) and big (Panels B and D) stock by variance risk premium (VRP, Panels A and B) and by covariance risk premium (CRP, Panels C and D), from simulated monthly data, at horizons $h = 1, 6,$ and 12 months, for different values of the initial consumption share of the pessimistic agent, $c_A = 0.1, 0.5,$ and 0.9. The small (big) stock has an initial dividend share of $s = 0.1$ ($s = 0.9$). The table shows the average of the regression coefficient and adjusted $R^2$ over all simulations, with standard errors in parenthesis. Returns are in annualized percentage while VRP and CRP are in monthly squared percentage.**

<table>
<thead>
<tr>
<th>Panel A: Regression of small stock returns on VRP</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c^A = 0.1$</td>
</tr>
<tr>
<td>Horizon (months)</td>
</tr>
<tr>
<td>VRP Coef</td>
</tr>
<tr>
<td>Adj R$^2$ (%)</td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Panel B: Regression of big stock returns on VRP</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c^A = 0.1$</td>
</tr>
<tr>
<td>Horizon (months)</td>
</tr>
<tr>
<td>VRP Coef</td>
</tr>
<tr>
<td>Adj R$^2$ (%)</td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Panel C: Regression of small stock returns on CRP</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c^A = 0.1$</td>
</tr>
<tr>
<td>Horizon (months)</td>
</tr>
<tr>
<td>CRP Coef</td>
</tr>
<tr>
<td>Adj R$^2$ (%)</td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Panel D: Regression of big stock returns on CRP</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c^A = 0.1$</td>
</tr>
<tr>
<td>Horizon (months)</td>
</tr>
<tr>
<td>CRP Coef</td>
</tr>
<tr>
<td>Adj R$^2$ (%)</td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td></td>
</tr>
</tbody>
</table>
Table 5: Survival. This table displays the share of consumption of the optimistic agent, \( c^B \), at horizon \( T = 50, 100, \) and \( 500 \) years, obtained from 1,000 simulations starting from \( c^B = 0.1, 0.5, \) and \( 0.9 \).

<table>
<thead>
<tr>
<th>( c^B )</th>
<th>( T = 50 )</th>
<th>( T = 100 )</th>
<th>( T = 500 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.08</td>
<td>0.07</td>
<td>0.02</td>
</tr>
<tr>
<td></td>
<td>(0.03)</td>
<td>(0.04)</td>
<td>(0.04)</td>
</tr>
<tr>
<td>0.5</td>
<td>0.44</td>
<td>0.38</td>
<td>0.14</td>
</tr>
<tr>
<td></td>
<td>(0.12)</td>
<td>(0.15)</td>
<td>(0.16)</td>
</tr>
<tr>
<td>0.9</td>
<td>0.86</td>
<td>0.81</td>
<td>0.44</td>
</tr>
<tr>
<td></td>
<td>(0.08)</td>
<td>(0.13)</td>
<td>(0.28)</td>
</tr>
</tbody>
</table>

Table 6: Market return predictability by variance risk premium. Data are monthly from January 1990 to December 2011.

<table>
<thead>
<tr>
<th></th>
<th>Panel A: OLS</th>
<th></th>
<th>Panel B: Robust regression</th>
</tr>
</thead>
<tbody>
<tr>
<td>Horizon (months)</td>
<td>1 3 6 9 12</td>
<td></td>
<td></td>
</tr>
<tr>
<td>VRP Coeff</td>
<td>-0.553 -0.555 -0.476</td>
<td>-0.824 -0.649 -0.473</td>
<td></td>
</tr>
<tr>
<td><code>t-stat</code></td>
<td>-2.048 -2.777 -3.705</td>
<td>-4.021 -5.896 -5.338</td>
<td></td>
</tr>
<tr>
<td>Adj ( R^2 ) (%)</td>
<td>1.88 5.99 7.75 5.74 4.63</td>
<td>1.71 6.16 8.11 6.10 4.98</td>
<td></td>
</tr>
</tbody>
</table>
Table 7: Return predictability by variance risk premium for CRSP cap-based portfolios. Panel A include regression estimates at the one-month horizon, Panel B is for the six-month horizon and Panel C for the 12-month horizon.

<table>
<thead>
<tr>
<th>Panel A: one-month horizon</th>
<th>Portfolio</th>
<th>Big Cap</th>
<th>Mid Cap</th>
<th>Small Cap</th>
</tr>
</thead>
<tbody>
<tr>
<td>VRP Coeff</td>
<td>-0.527</td>
<td>-0.597</td>
<td>-0.630</td>
<td></td>
</tr>
<tr>
<td>t-stat</td>
<td>-1.975</td>
<td>-2.075</td>
<td>-1.820</td>
<td></td>
</tr>
<tr>
<td>Adj $R^2$ (%)</td>
<td>1.69</td>
<td>1.45</td>
<td>1.12</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Panel B: six-month horizon</th>
<th>Portfolio</th>
<th>Big Cap</th>
<th>Mid Cap</th>
<th>Small Cap</th>
</tr>
</thead>
<tbody>
<tr>
<td>VRP Coeff</td>
<td>-0.435</td>
<td>-0.573</td>
<td>-0.736</td>
<td></td>
</tr>
<tr>
<td>t-stat</td>
<td>-3.335</td>
<td>-3.844</td>
<td>-3.720</td>
<td></td>
</tr>
<tr>
<td>Adj $R^2$ (%)</td>
<td>6.21</td>
<td>8.19</td>
<td>10.17</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Panel C: 12-month horizon</th>
<th>Portfolio</th>
<th>Big Cap</th>
<th>Mid Cap</th>
<th>Small Cap</th>
</tr>
</thead>
<tbody>
<tr>
<td>VRP Coeff</td>
<td>-0.229</td>
<td>-0.354</td>
<td>-0.450</td>
<td></td>
</tr>
<tr>
<td>t-stat</td>
<td>-2.310</td>
<td>-3.099</td>
<td>-3.237</td>
<td></td>
</tr>
<tr>
<td>Adj $R^2$ (%)</td>
<td>2.86</td>
<td>7.28</td>
<td>9.67</td>
<td></td>
</tr>
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</table>
Table 8: Return predictability by variance risk premium for small-, mid-, and big-cap portfolios from CRSP, at the 6-month horizon, for different levels of the difference in beliefs. For comparison, the last two panel report results of the same predictive regression for the S&P500 index return and for small- minus big-cap portfolio return, respectively.

<table>
<thead>
<tr>
<th></th>
<th>Small DB</th>
<th>Average DB</th>
<th>Large DB</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>small-cap</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>VRP Coeff</td>
<td>0.06</td>
<td>-0.32</td>
<td>-1.37</td>
</tr>
<tr>
<td>Adj $R^2$ (%)</td>
<td>-1.36</td>
<td>2.87</td>
<td>21.26</td>
</tr>
<tr>
<td><strong>mid-cap</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>VRP Coeff</td>
<td>0.19</td>
<td>-0.29</td>
<td>-1.11</td>
</tr>
<tr>
<td>Adj $R^2$ (%)</td>
<td>-0.41</td>
<td>3.82</td>
<td>17.79</td>
</tr>
<tr>
<td><strong>big-cap</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>VRP Coeff</td>
<td>-0.18</td>
<td>-0.29</td>
<td>-0.82</td>
</tr>
<tr>
<td>Adj $R^2$ (%)</td>
<td>-0.04</td>
<td>2.86</td>
<td>14.41</td>
</tr>
<tr>
<td><strong>S&amp;P500</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>VRP Coeff</td>
<td>-0.24</td>
<td>-0.31</td>
<td>-0.97</td>
</tr>
<tr>
<td>Adj $R^2$ (%)</td>
<td>1.09</td>
<td>4.02</td>
<td>20.91</td>
</tr>
</tbody>
</table>

Table 9: Mean and standard deviation of returns (in annualized percentage) for small- and big-cap portfolios from CRSP, at the 6-month horizon, for different levels of the difference in beliefs.

<table>
<thead>
<tr>
<th></th>
<th>Small DB</th>
<th>Average DB</th>
<th>Large DB</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>small-cap</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mean (%)</td>
<td>5.40</td>
<td>18.17</td>
<td>11.37</td>
</tr>
<tr>
<td>Standard Deviation (%)</td>
<td>21.54</td>
<td>22.90</td>
<td>45.48</td>
</tr>
<tr>
<td><strong>big-cap</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mean (%)</td>
<td>7.94</td>
<td>14.53</td>
<td>2.04</td>
</tr>
<tr>
<td>Standard Deviation (%)</td>
<td>15.00</td>
<td>19.62</td>
<td>32.61</td>
</tr>
</tbody>
</table>
Figure 1: Instantaneous equity premium of stock 1, under agent A’s beliefs, as a function of the dividend share of asset 1, $s_1$, when the consumption share of the pessimistic agent is $c^A = 0.1$ (left panel) or $c^A = 0.9$ (right panel).
Figure 2: Instantaneous variance risk premium of stock 1, under agent A’s beliefs, in monthly squared percentage, as a function of the dividend share of asset 1, $s_1$, for different values of the consumption share of the pessimistic agent is $c^A$. The second, third and fourth panels show the decomposition of the individual variance risk premium in its idiosyncratic and systematic jump components when the consumption share of the pessimistic agent is $c^A = 0.1$, $c^A = 0.5$ and $c^A = 0.9$, respectively.
Figure 3: Instantaneous equity (upper panels, in percentage) and variance (lower panels, in monthly squared percentage) risk premium of the market, under agent A’s beliefs, as a function of the dividend share of asset 1, $s_1$, and their decomposition in terms of individual equity and variance premia, for different values of the consumption share of the pessimistic agent is $c_A$. The first and third panels use $c_A = 0.1$, while the second and the fourth are for $c_A = 0.9$. 
Figure 4: Instantaneous conditional stock return correlation under the physical and risk-neutral measure, and instantaneous correlation risk premium, in an economy with \( N = 2 \) assets.
**Figure 5:** Instantaneous equity premium (annualized and in percentage, first panel), systematic jump premium (second panel) and index variance risk premium (in monthly terms and squared percentage, third panel), under agent A’s beliefs, in the case of a large symmetric economy, as a function of the consumption share of the pessimistic agent, $c^A$. 

![Graph showing equity premium, jump premium, and variance premium as functions of $c^A$.]
Figure 6: Time series of variance risk premium, in monthly squared percentage, where the physical expectation of the realized variance is computed from a projection of realized variance on the value of the lagged squared VIX and on lagged realized variance. Light gray shaded areas denote phases in which difference in beliefs, measured based on the dispersion of one-year-ahead forecasts on real GDP growth from the BlueChip Economic Indicator, is above average. Dark gray shaded areas denote NBER recessions.
**Figure 7:** Standard OLS regression of excess market returns at the six-month horizon on the lagged variance risk premium, for different levels of the VRP. The first box plot corresponds to small absolute values of the premium (VRP < q$_{70\%}$), the last to large values (VRP > q$_{30\%}$) and the middle box plot to average values of the VRP. Upper panel display the distribution of regression coefficients and lower panel of percentage $R^2$, both obtained applying a block bootstrap procedure.

![Box plots of regression coefficients and $R^2$](image)

**Figure 8:** Kernel regression of standardized excess market returns at the six-month horizon, in annualized percentage, on the lagged variance risk premium, in monthly squared percentage. Single dots represent the data, while the solid line is an estimated kernel regression using Nadaraya–Watson estimator with a Gaussian kernel.

![Kernel regression plot](image)
**Figure 9:** Predictive regressions of market 6-month excess returns on lagged variance risk premium on a rolling window of 50 months. Upper panel shows regression coefficient estimates with 95% confidence bounds, while lower panel reports adjusted $R^2$ in percentage. The dashed red line in the upper panel denotes the regression coefficient estimated on the full sample.

**Figure 10:** Standard OLS regression of excess returns of Fama and French portfolios at the six-month horizon on the lagged market variance risk premium. Left panel displays the regression coefficients and right panel the percentage $R^2$. Lines connect portfolios of different book-to-market categories within each size category, focusing on the bottom and upper quintiles, which correspond to small and big stocks, respectively.