Supplemental Appendix

for

Heterogeneous Beliefs about Rare Event Risk in the Lucas Orchard

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This Supplemental Appendix provides all the proofs of the propositions in the paper in Section A, while Section B shows the results of the predictive regressions in Section 4 of the main text using Bollerslev and Todorov (2011)’s time series of market variance risk premium only due to large jumps.

A Proofs

A.1 Proof of Proposition 1

The equilibrium allocations can be characterized by solving the optimization problem of a representative agent whose utility function is a weighted sum of the two agents’ utilities,

\[ U(C(t), \phi(t)) = \max_{C^A + C^B = C} \{ U^A(C^A(t)) + \phi(t)U^B(C^B(t)) \} . \]  

(1)

Given multiplier \( \xi \) for the constraint in representative agent problem (1),

\[ U^A(C^A(t))' = \xi = \phi(t)U^B(C^B(t))' \Rightarrow \phi(t) = \frac{U^A(C^A(t))'}{U^B(C^B(t))'} = \frac{y^A\eta^A(t)}{y^B\eta^B(t)} , \]  

(2)
where the last equality follows from individual agents optimality. Equation (2) and the optimality condition of agent $A$ imply:

$$U(C(t), \phi(t)) = U^A(C^A(t)) + \phi U^B(C^B(t)) = U^A(C^A(t)) = y^A e^{\delta t} \eta^A(t).$$

Therefore, individual agents optimal consumptions can be written as

$$C^A(t) = (y^A e^{\delta t} \eta^A(t))^{-1/\gamma} = U''(C(t), \phi(t))^{-1/\gamma}, \quad (4)$$

$$C^B(t) = (y^B e^{\delta t} \eta^B(t))^{-1/\gamma} = \left(\frac{y^A e^{\delta t} \eta^A(t)}{\phi(t)}\right)^{-1/\gamma}.$$  

Using the market clearing condition $C(t) = C^A(t) + C^B(t)$,

$$C(t) = U''(C(t), \phi(t))^{-1/\gamma} + \left(\frac{U''(C(t), \phi(t))}{\phi(t)}\right)^{-1/\gamma},$$

which can be solved for the marginal utility of the representative agent, leading to

$$U''(C(t), \phi(t)) = \frac{(1 + \phi(t)^{(1/\gamma)})^\gamma}{C(t)^\gamma}. \quad (6)$$

Inserting (6) in (4) and (5) lead to the equilibrium consumption allocations:

$$C^A(t) = \frac{1}{1 + \phi(t)^{(1/\gamma)}} C(t) \quad \text{and} \quad C^B(t) = \frac{\phi(t)^{(1/\gamma)}}{1 + \phi(t)^{(1/\gamma)}} C(t),$$  

and investors’ state price densities:

$$\eta^A(t) = e^{-\delta t} \frac{(1 + \phi(t)^{(1/\gamma)})^\gamma}{y^A C(t)^\gamma} \quad \text{and} \quad \eta^B(t) = \eta^A(t) \frac{\phi(0)}{\phi(t)} = e^{-\delta t} \frac{(1 + \phi(t)^{(1/\gamma)})^\gamma}{y^B C(t)^\gamma} \phi(t). \quad (8)$$

Here $\phi(0)$ solves either agent’s individual budget constraint, and the stochastic weighting process $\phi(t) = y^A \eta^A(t)/y^B \eta^B(t)$ follows the dynamics given in Equation (4) in the main text with jump intensity $\lambda_c^A(t)$.

### A.2 Proof of Proposition 2

The state price density of agent $A$ is given in Proposition 1:

$$\eta^A(t) = e^{-\delta t} \frac{(1 + \phi(t)^{(1/\gamma)})^\gamma}{y^A C(t)^\gamma} \quad (8)$$

---

$^1$The budget constraints of agents determine only the ratio $y^A/y^B$. I set $y^A = U''(C(0), \phi(0))$ without loss of generality, so that $\eta^A(0) = \eta^B(0) = 1$. 

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Applying Ito’s lemma and using the dynamics of \( \phi(t) \) and \( C(t) \), the dynamics of \( \eta^A \) is given by:

\[
d\eta^A(t) = -\delta \eta^A(t) dt + \eta^A(t) \frac{\phi(t)^{1/\gamma}}{1 + \phi(t)^{1/\gamma}} (\beta^A - \beta^B) X(t) dt - \gamma \eta^A(t) (\mu dt + \sigma \sum_{j=1}^{N} s_j dW_{jt}) + \\
\frac{1}{2} \gamma (\gamma + 1) \eta^A(t) \sigma^2 \sum_{j=1}^{N} s_j^2 dt + \eta^A(t) \sum_{j=1}^{N} [(s_j k + 1)^{-\gamma} - 1] dN_{jt} + \\
\eta^A(t) \left[ \left( 1 + \frac{\phi(t) \beta^B}{\beta^A} \right)^{1/\gamma} \right] (k + 1)^{-\gamma} - 1 \right] dN_{ct}. \tag{9}
\]

Therefore,

\[
\frac{d\eta^A(t)}{\eta^A(t)} = \left[ -\delta + \frac{\phi(t)^{1/\gamma}}{1 + \phi(t)^{1/\gamma}} (\beta^A - \beta^B) X(t) - \gamma \mu + \frac{1}{2} \gamma (\gamma + 1) \sigma^2 \sum_{j=1}^{N} s_j^2 \right] dt - \gamma \sigma \sum_{j=1}^{N} s_j dW_{jt} + \\
\sum_{j=1}^{N} [(s_j k + 1)^{-\gamma} - 1] dN_{jt} + \left[ \left( 1 + \frac{\phi(t) \beta^B}{\beta^A} \right)^{1/\gamma} \right] (k + 1)^{-\gamma} - 1 \right] dN_{ct}. \tag{10}
\]

Comparing the drift, diffusion and jump terms of this expression with those in equation (11) in the main text directly leads to the solution for \( \theta_j, \lambda^Q_j, \lambda^Q_c \) and \( r \) in Proposition 2.

### A.3 Proof of Proposition 3

Asset prices will depend on the \( N-1 \) dividend shares, but following Martin (2013) I introduce a monotonic transformation of these state variables,

\[
u_j = \ln \frac{s_j}{s_1}, \tag{11}\]

which measures the size of asset \( j \) relative to asset 1. As \( s_j \) ranges from 0 to 1, \( u_j \) can take all values on the real line. Applying Itô’s lemma to the definition in Equation (11) while assuming symmetric assets, we obtain the dynamics

\[
du_j = d \ln D_j - d \ln D_1 = \sigma (dW_{jt} - dW_{1t}) + \ln(k + 1)(dN_{jt} - dN_{1t}) \tag{12}\]

for \( j = 2, \ldots, N \). In matrix notation, the dynamics of \( u = (u_2, \ldots, u_N)' \) is given by

\[
du = \sigma UdW_t + \ln(k + 1) UdN_t, \tag{13}\]

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where $U$ is a $(N - 1) \times N$ matrix:

$$
U \equiv \begin{pmatrix}
-1 & 1 & 0 & \cdots & 0 \\
-1 & 0 & 1 & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & 0 \\
-1 & 0 & \cdots & 0 & 1
\end{pmatrix}.
$$

For simplicity, I first provide the derivation of the stock price expression for the case of a two-trees economy, to then extend it to the case of $N$ stocks.

### A.3.1 Stock prices in an economy with $N = 2$ stocks

For notational convenience, I compute the price of stock 1, the price of stock 2 follows an analogous expression with obvious modifications. Let me define $y_{it} \equiv \ln D_i(t)$ and $\tilde{y}_{i\tau} \equiv y_{i\tau} - y_{it}$. The price of stock 1 is given by the discounted value of all its future dividends:

$$S_1(t) = E_t^A \left[ \int_t^T \frac{\eta^A(s)}{\eta^A(t)} D_1(s) ds \right]. \quad (14)$$

This can also be viewed as a portfolio of zero coupon dividend claims:

$$S_1(t) = \int_t^T S^*_1(t)d\tau$$

with

$$S^*_1(t) = E_t^A \left[ \frac{\eta^A(\tau)}{\eta^A(t)} D_1(\tau) \right]$$

$$= e^{-\delta(\tau-t)} C(t) \gamma \frac{1 + \phi(t)^{1/\gamma}}{(1 + \phi(t)^{1/\gamma})^\gamma} E_t^A \left[ \sum_{k=0}^\gamma \left( \frac{\gamma}{k} \right) \phi(\tau)^{k/\gamma} e^{y_{1t} + \tilde{y}_{1\tau}} \right]$$

$$= e^{-\delta(\tau-t)} C(t) \gamma \frac{1 + \phi(t)^{1/\gamma}}{(1 + \phi(t)^{1/\gamma})^\gamma} E_t^A \left[ \sum_{k=0}^\gamma \left( \frac{\gamma}{k} \right) \phi(\tau)^{k/\gamma} e^{(1-\gamma/2) y_{1t} - \gamma/2 y_{2t} + (1-\gamma/2) \tilde{y}_{1\tau} - \gamma/2 \tilde{y}_{2\tau}} \right]$$

$$= e^{-\delta(\tau-t)} C(t) \gamma \frac{1 + \phi(t)^{1/\gamma}}{(1 + \phi(t)^{1/\gamma})^\gamma} e^{(1-\gamma/2) y_{1t} - \gamma/2 y_{2t}} \sum_{k=0}^\gamma \left( \frac{\gamma}{k} \right) E_t^A \left[ \frac{e^{k/\gamma \ln \phi(\tau) + (1-\gamma/2) \tilde{y}_{1\tau} - \gamma/2 \tilde{y}_{2\tau}}}{(2 \cosh \left( y_{2t} y_{1t} + \tilde{y}_{1\tau} - \tilde{y}_{2\tau} \right))^{\gamma}} \right]$$

assuming an integer coefficient of relative risk aversion $\gamma$. Then I use the fact that

$$\frac{1}{[2 \cosh(u/2)]^{\gamma}} = \int_{-\infty}^{\infty} e^{iuz} F_\gamma(z) dz,$$
where the Fourier transform $\mathcal{F}_y(z)$ is given by (see Martin (2013)):

$$
\mathcal{F}_y(z) \equiv \frac{\Gamma(\gamma/2 + iz)\Gamma(\gamma/2 - iz)}{2\pi\Gamma(\gamma)}.
$$  \hspace{1cm} (16)

The conditional expectation in Equation (15) can thus be written as

$$
E_t^A \left[ \frac{e^{k/\gamma \ln(\phi(x) + (1-\gamma/2)\tilde{y}_{ir} - \gamma/2\tilde{y}_{tr})}}{(2 \cosh (\tilde{y}_{ir} - \tilde{y}_{tr} + \tilde{y}_{tr} - \tilde{y}_{ir}))^\gamma} \right] = \int_{-\infty}^{\infty} \mathcal{F}_y(z)e^{iz(y_{ir} - y_{tr})}E_t^A \left[ \frac{e^{k/\gamma \ln(\phi(x) + (1-\gamma/2-iz)\tilde{y}_{ir} + (\gamma/2+iz)\tilde{y}_{tr})}}{(2 \cosh (\tilde{y}_{ir} - \tilde{y}_{tr} + \tilde{y}_{tr} - \tilde{y}_{ir}))^\gamma} \right] dz
$$

$$
= \int_{-\infty}^{\infty} \mathcal{F}_y(z)e^{iz(y_{ir} - y_{tr})}D_1(t)^{-(1-\gamma/2-iz)}D_2(t)^{-(iz-\gamma/2)}e^{(\tau-t)(\mu - 1/2\sigma^2)+1/2\sigma^2((1-\gamma/2-iz)^2+(\gamma/2+iz)^2)+(1-\gamma/2-iz)y_{tr}^0+(\gamma/2+iz)y_{ir}^0}E_t^A \left[ e^{k/\gamma \ln(\phi(x) + (1-\gamma/2-iz)y_{ir}^0(t) + (\gamma/2+iz)y_{tr}^0(t))} \right] dz,
$$  \hspace{1cm} (17)

where $y_{ir}^c(t)$ and $y_{tr}^d(t)$, for $i = 1, 2$ are the diffusion and jump components of log dividends and their dynamics are given by:

$$
dy_{ir}^c(t) = \left( \mu - \frac{1}{2}\sigma^2 \right) dt + \sigma dW_{it},
$$  \hspace{1cm} (18)

$$
dy_{tr}^d(t) = \ln (k + 1) (dN_{it} + dN_{ct}),
$$  \hspace{1cm} (19)

and

$$
d\ln \phi(t) = (\beta^A - \beta^B)X(t)dt + \ln \left( \frac{\beta^B}{\beta^A} \right) dN_{ct},
$$  \hspace{1cm} (20)

from equation (4) in the main text.

Thus, denoting $x(t) \equiv \frac{k}{\gamma} \ln(\phi(x) + (1-\gamma/2-iz)y_{ir}^d(\tau) + (\gamma/2+iz)y_{tr}^d(\tau))$, its dynamics follows:

$$
dx(t) = \frac{k}{\gamma}(\beta^A - \beta^B)X(t)dt + (1-\gamma/2-iz)\ln(k + 1) dN_{1t} + (iz - \gamma/2)\ln(k + 1) dN_{2t}
$$

$$
+ \left[ (1-\gamma)\ln(k+1) + \frac{k}{\gamma} \ln \left( \frac{\beta^B}{\beta^A} \right) \right] dN_{ct},
$$  \hspace{1cm} (21)

and also depends on the state variable $X(t)$. Let me define $Y = \begin{bmatrix} x \\ X \end{bmatrix}$, whose dynamics can be written as

$$
dY = \left[ \begin{array}{cc} 0 & \frac{k}{\gamma}(\beta^A - \beta^B) \\ \varphi & -\varphi \end{array} \right] dt + \left[ \begin{array}{cc} 0 & \frac{k^1}{\gamma} \\ \sigma X \sqrt{X} & 0 \end{array} \right] dW_t^X + \left[ \begin{array}{cc} k_1^X & k_2^X \\ 0 & 0 \end{array} \right] dN_{1t} + \left[ \begin{array}{cc} k_1^c & k_2^c \\ 0 & 0 \end{array} \right] dN_{ct},
$$  \hspace{1cm} (22)
with jump intensities \( \lambda_Y^1 = \lambda_Y^2 = \lambda \) and \( \lambda_Y^c = [0 \quad \beta^A]Y \).

The conditional expectation in equation (17) can thus be written as

\[
f(Y, t) = E^A_t [e^{wY}],
\]

with \( w = [1 \quad 0] \) and since \( Y \) follows an affine jump diffusion we know (see Duffie, Pan, and Singleton (2000)) that \( f(Y, t) \) is of the form \( f(Y, t) = e^{\alpha_0(k(s) + \alpha_1(k(s) \tau(t) + \alpha_2(k(s))X(t))} \), where \( s = \tau - t \) and \( \alpha_0(k(s), \alpha_1(k(s) \alpha_2(k(s)) \) follow the system of Riccati equations:

\[
\begin{align*}
\alpha'_{0,k}(s) &= \alpha_{2,k}(s)\varphi + \lambda(e^{k^2} - 1 + e^{k^2} - 1), \\
\alpha'_{1,k}(s) &= 0, \\
\alpha'_{2,k}(s) &= \frac{k}{\gamma}(\beta^A - \beta^B)\alpha_{1,k}(s) - \varphi\alpha_{2,k}(s) + \frac{1}{2}\sigma^2\alpha_{2,k}(s)^2 + \beta^A(e^{k^2} - 1),
\end{align*}
\]  

with initial conditions, \( \alpha_{0,k}(0) = \alpha_{2,k}(0) = 0 \) and \( \alpha_{1,k}(0) = 1. \)

From (24) we find \( \alpha_{1,k}(s) = \alpha_{1,k}(0) = 1, \) while \( \alpha_{0,k}(s) \) and \( \alpha_{2,k}(0) \) are easily solved numerically.\(^2\)

Therefore, \( f(Y, t) = e^{\alpha_0(k(s) + \tau(t) + \alpha_2(k(s))X(t)} \), and the price of a zero coupon dividend claim is

\[
S_1^*(t) = e^{-\delta(\tau-t)} \left( \frac{C(t)\gamma}{(1 + \phi(t))^{1/\gamma}} \right) \frac{D_1(t)}{D_2(t)} \sum_{k=0}^{\gamma} \left( \frac{\gamma}{k} \right) \phi(t)^{k/\gamma} \int_{-\infty}^{\infty} \mathcal{F}_\gamma(z)e^{iz(y_2 - y_1)} dz 
\]

\[
\cdot e^{(\tau-t)(1-\gamma)(\mu-1/2\sigma^2)+1/2\sigma^2((1-\gamma)(1-\gamma) - 2(1-\gamma)^2) + (1-\gamma)(1-\gamma)^2) + \alpha_{0,k}(\gamma-\tau) + \alpha_{2,k}(t)X(t) d\tau} 
\]

\[
\]  

Therefore the price of the aggregate consumption claim is

\[
S_1(t) = \int_t^T S_1^*(t) d\tau 
\]

\[
= D_1(t) \left[ 2 \cosh \left( \frac{\mu t}{2} \right) \right] \sum_{k=0}^{\gamma} a_k(\phi) \int_{-\infty}^{\infty} \mathcal{F}_\gamma(z)e^{izu} b_k(X, t, z) dz,
\]

where

\[
a_k(\phi) = \left( \frac{\gamma}{k} \right) \frac{\phi(t)^{k/\gamma}}{(1 + \phi(t))^{1/\gamma}} \gamma, \]  

\[
b_k(X, t) = \int_t^T e^{(\tau-t)(1-\gamma)(\mu-1/2\sigma^2)+1/2\sigma^2((1-\gamma)(1-\gamma) - 2(1-\gamma)^2) + (1-\gamma)(1-\gamma)^2) + \alpha_{0,k}(\gamma-\tau) + \alpha_{2,k}(t)X(t) d\tau},
\]

\[
\]  

\(^2\)In my simulations I use a Runge-Kutta 4th order method.
and the price-dividend ratio of stock 1 is

\[ g_1(\phi, X, u, t) \equiv \frac{S_1(t)}{D_1(t)} = \left[ 2 \cosh \left( \frac{u t}{2} \right) \right] ^\gamma \sum_{k=0}^{\gamma} a_k(\phi) \int_{-\infty}^{\infty} F_\gamma(z) e^{izut} b_k(X, t, z) dz. \]

In the same way it is possible to obtain the closed form expression for the price of stock 2, \( S_2(t) \).

### A.3.2 General stock price expressions in an economy with \( N \) stocks

The basic approach is the same with \( N > 2 \) assets. The main technical difficulty lies in generalizing \( F_\gamma(z) \) to the \( N \)-asset case, but this problem is solved by Martin (2013), who defines

\[ F_N^\gamma(z) \equiv \frac{\Gamma(\gamma/N + iz_1 + iz_2 + \ldots + iz_{N-1})}{(2\pi)^{N-1} \Gamma(\gamma)} \prod_{k=1}^{N-1} \Gamma(\gamma/N - iz_k). \]  

(30)

The price-dividend ratio on an asset \( j \) is thus:

\[ g_j(\phi, X, u, t) = e^{-\gamma \sum_{j=2}^{N} u_j/N} (1 + e^{u_2} + \ldots + e^{u_N})^\gamma \sum_{k=0}^{\gamma} a_k(\phi) \int F_N^\gamma(z) e^{izu_j} b_{jk}(X, t, z) dz, \]

(31)

where the integral is evaluated on \( \mathbb{R}^{N-1} \), \( a_k(\phi) \) is given in Equation (28) and \( b_{jk}(X, t, z) \) generalizes Equation (29) as follows:

\[ b_{jk}(X, t, z) = \int_t^T e^{(\tau-t)[\delta + (\mu - \frac{1}{2} \sigma^2)\tau]} 1_N(e_j - \gamma/N + \mu U'z + \frac{1}{2} \sigma^2(e_j - \gamma/N + iU'z})^\gamma + a_{N,0,k}(\tau-t) + a_{N,2,k}(\tau-t)X(t) d\tau, \]

where \( e_j \) is the \( N \)-vector with a 1 at the \( j \)-th entry and zeros elsewhere, \( 1_N \) is a \( N \)-dimensional vector of ones and \( a_{N,k}(\tau) \) and \( \alpha_{N,k}(\tau) \) satisfy the following system of Riccati equations:

\[
\alpha_{0,k}^N(s)' = \alpha_{2,k}^N(s) \varphi + \lambda \sum_{i=1}^{N} (e_{k,i} - 1), \\
\alpha_{2,k}^N(s)' = \frac{k}{\gamma} (\beta A - \beta B) - \varphi \alpha_{2,k}^N(s) + \frac{1}{2} \sigma^2 \alpha_{2,k}^N(s)^2 + \beta A (e_k^N - 1),
\]

with initial conditions, \( \alpha_{0,k}^N(0) = \alpha_{2,k}^N(0) = 0. \)

### A.3.3 The case of a large economy: \( N \rightarrow \infty \)

In the case of a large diversified economy considered in Section 3.4 of the main text, i.e. \( N \) large and \( s_j = 1/N \), price expressions simplify since \( u_j = 0 \ \forall j \). The price-dividend
ratio of any stock $j$ is given by:

$$g_j(\phi, X, u, t) = N^\gamma \sum_{k=0}^{\gamma} a_k(\phi) \int \mathcal{F}_\gamma^N(z) b_{jk}(X, t, z) dz. \quad (32)$$

### B Variance risk premium due to large jumps

Bollerslev and Todorov (2011) develop a nonparametric method to isolate the fraction of the observed variance risk premium due to large jumps. Since the model-implied variance risk premium only includes compensation for jump risk, it is useful, as a robustness check, to run the predictive regressions in Section 4 of the main text using Bollerslev and Todorov (2011)’s time series of market variance risk premium only due to large jumps ($VRP^j$), which is available from February 1996 through July 2007. Figure I compares this measure of the variance premium only due to jumps with the variance risk premium measure used in the main text. The correlation between the two series is about 73%. I first consider the predictive regression

$$r_{t+6}^e = \alpha + \beta VRP^j_t + \varepsilon_{t+6}, \quad (33)$$

where $r_{t+6}^e$ is the excess return of the S&P500 index at a 6 months horizon. Figure II shows regression coefficient estimates with 95% confidence bounds (upper panel) and adjusted $R^2$ in percentage (upper panel) estimated on a rolling window of 50 months. Consistent with the results in the main text, predictive power is stronger in phases of large disagreement, such as in the early 2000 and at the onset of the recent financial crisis.

Then Figure III shows the distributions of regression coefficients (upper panel) and $R^2$ (lower panel), for small, average, and large values of the difference in beliefs, obtained applying a block bootstrap procedure. Both the regression coefficient and the adjusted $R^2$ increase (in absolute value) with the level of DB. Results are a bit less strong than what I find using the aggregate variance risk premium in the main text, but they have to be taken with caution since the number of observations in every bin is small.$^3$

Then I estimate regressions of the form:

$$r_{i,t+6}^e = \alpha_i + \beta_i VRP^j_t + \varepsilon_{t+6}, \quad (34)$$

$^3$Since $VRP^j$ is available only from February 1996 through July 2007, there are less than 50 monthly observations for each DB quantile.
for \( i = S, M, B \), where \( r_{i,t+h}^{c} \) is the monthly excess returns on small-, mid-, and big-cap portfolios, respectively, at the 6-month horizon. Consistent with the results in the main text, predictive power is stronger for small stocks, with an adjusted \( R^2 \) of 2.1% against an adjusted \( R^2 \) of −0.4% for large stocks.
C Figures

Figure I: Time series of variance risk premium only due to jumps, from Bollerslev and Todorov (2011), versus the total variance risk premium measure described in Section 4 of the main text, for the overlapping sample, that goes from February 1996 through July 2007. Both measures are in monthly squared percentage.
**Figure II:** Predictive regressions of market 6-month excess returns on lagged variance risk premium only due to jumps, from Bollerslev and Todorov (2011), on a rolling window of 50 months. Upper panel shows regression coefficient estimates with 95% confidence bounds, while lower panel reports adjusted $R^2$ in percentage.
Figure III: Standard OLS regression of excess market returns at the six-month horizon on the lagged variance risk premium only due to jumps, from Bollerslev and Todorov (2011), for different levels of the difference in beliefs (DB). The first box plot corresponds to small values of disagreement (DB $< q_{30\%}$), the last to large values (DB $> q_{70\%}$) and the middle box plot to average values of DB. Upper panel display the distribution of regression coefficients and lower panel of percentage $R^2$, both obtained applying a block bootstrap procedure.
References

