

Fostering Entrepreneurship: Promoting Founding or Funding?

Thomas Hellmann and Veikko Thiele

— ONLINE APPENDIX —

January 2017

1 Optimal Investment for External Angels

External angels choose k to maximize $U_I(k)$, where $\delta(1+r) = 1$. Using the first-order condition for k^* we find that $\theta = \alpha \frac{\delta \rho y}{\phi} - 1$. Moreover, note that $U_I(k)$ is linear in k . Thus, we get a bang-bang solution where it is optimal for external angels to invest their entire wealth in new ventures ($k^* = \tilde{w}$) when $\theta \leq \hat{\theta} \equiv \alpha \frac{\delta \rho y}{\phi} - 1$, and to invest their entire wealth in the safe asset ($k^* = 0$) when $\theta > \hat{\theta}$.

2 Benchmark Model: Market Equilibrium

Noting that l is uniformly distributed with support $[0, \mu_E]$ we can write the entry condition for entrepreneurs as $n_E = \frac{1}{\mu_E} U_E = \frac{1}{\mu_E} \delta \rho (1 - \alpha) y$. Moreover, because θ is uniformly distributed with support $[0, \mu_I]$, we can write the market clearing condition as

$$\frac{1}{\mu_E} \delta \rho (1 - \alpha) y \phi = \frac{1}{\mu_I} \left[\alpha \frac{\delta \rho y}{\phi} - 1 \right] \tilde{n} \tilde{w}, \quad (\text{A.1})$$

which defines the equilibrium equity stake for investors, α^* , with $\alpha^* \in \left[\frac{\phi}{\delta \rho y}, 1 \right]$. The total capital demand, $E(\alpha) \equiv \frac{1}{\mu_E} \delta \rho (1 - \alpha) y \phi$, is decreasing in α , while the total capital supply from investors, $I(\alpha) \equiv \frac{1}{\mu_I} \left[\alpha \frac{\delta \rho y}{\phi} - 1 \right] \tilde{n} \tilde{w}$, is increasing in α . Moreover, note that $E\left(\frac{\phi}{\delta \rho y}\right) > I\left(\frac{\phi}{\delta \rho y}\right) = 0$, and $I(1) > E(1) = 0$. Thus, the market equilibrium is unique.

Implicitly differentiating (A.1) we find

$$\begin{aligned}\frac{d\alpha^*}{d\phi} &= \frac{\frac{1}{\mu_E}\delta\rho(1-\alpha)y + \frac{1}{\mu_I}\alpha\frac{\delta\rho y}{\phi^2}\tilde{n}\tilde{w}}{\frac{1}{\mu_E}\delta\rho y\phi + \frac{1}{\mu_I}\frac{\delta\rho y}{\phi}\tilde{n}\tilde{w}} > 0 & \frac{d\alpha^*}{d(\tilde{n}\tilde{w})} &= -\frac{\frac{1}{\mu_I}\left[\alpha\frac{\delta\rho y}{\phi} - 1\right]}{\frac{1}{\mu_E}\delta\rho y\phi + \frac{1}{\mu_I}\frac{\delta\rho y}{\phi}\tilde{n}\tilde{w}} < 0 \\ \frac{d\alpha^*}{d\mu_I} &= \frac{\frac{1}{\mu_I^2}\left[\alpha\frac{\delta\rho y}{\phi} - 1\right]\tilde{n}\tilde{w}}{\frac{1}{\mu_E}\delta\rho y\phi + \frac{1}{\mu_I}\frac{\delta\rho y}{\phi}\tilde{n}\tilde{w}} > 0 & \frac{d\alpha^*}{d\mu_E} &= -\frac{\frac{1}{\mu_E^2}\delta\rho(1-\alpha)y\phi}{\frac{1}{\mu_E}\delta\rho y\phi + \frac{1}{\mu_I}\frac{\delta\rho y}{\phi}\tilde{n}\tilde{w}} < 0.\end{aligned}$$

Likewise,

$$\frac{d\alpha^*}{d(\rho y)} = \frac{\overbrace{\frac{1}{\mu_E}\delta(1-\alpha)\phi - \frac{1}{\mu_I}\alpha\frac{\delta}{\phi}\tilde{n}\tilde{w}}^{\equiv Z}}{\frac{1}{\mu_E}\delta\rho y\phi + \frac{1}{\mu_I}\frac{\delta\rho y}{\phi}\tilde{n}\tilde{w}}.$$

Note that the market clearing condition (A.1) can be written as

$$\underbrace{\left[\frac{1}{\mu_E}\delta(1-\alpha)\phi - \frac{1}{\mu_I}\alpha\frac{\delta}{\phi}\tilde{n}\tilde{w}\right]}_{\equiv Z} = -\frac{1}{\mu_I}\tilde{n}\tilde{w}.$$

Thus, $Z < 0$ in equilibrium. Consequently, $d\alpha^*/d(\rho y) < 0$.

Finally, using the entry condition $n_E = \frac{1}{\mu_E}\delta\rho(1-\alpha)y$, we find

$$\begin{aligned}\frac{dn_E^*}{d\phi} &= -\frac{1}{\mu_E}\delta\rho y \underbrace{\frac{d\alpha^*}{d\phi}}_{>0} < 0 & \frac{dn_E^*}{d(\tilde{n}\tilde{w})} &= -\frac{1}{\mu_E}\delta\rho y \underbrace{\frac{d\alpha^*}{d(\tilde{n}\tilde{w})}}_{<0} > 0 \\ \frac{dn_E^*}{d\mu_I} &= -\frac{1}{\mu_E}\delta\rho y \underbrace{\frac{d\alpha^*}{d\mu_I}}_{>0} < 0 & \frac{dn_E^*}{d(\rho y)} &= \frac{1}{\mu_E}\delta(1-\alpha) - \frac{1}{\mu_E}\delta\rho y \underbrace{\frac{d\alpha^*}{d(\rho y)}}_{<0} > 0.\end{aligned}$$

Moreover, using the expression for $d\alpha^*/d\mu_E$ and simplifying, we get

$$\frac{dn_E^*}{d\mu_E} = -\frac{1}{\mu_E^2}\delta\rho(1-\alpha)y - \frac{1}{\mu_E}\delta\rho y \frac{d\alpha^*}{d\mu_E} = -\frac{\frac{1}{\mu_E^2}\delta\rho(1-\alpha)y\frac{1}{\mu_I}\frac{1}{\phi}\tilde{n}\tilde{w}}{\frac{1}{\mu_E}\phi + \frac{1}{\mu_I}\frac{1}{\phi}\tilde{n}\tilde{w}} < 0.$$

3 Proof of Proposition 1

To show that the market equilibrium is efficient, we derive the socially optimal (i.e., first best) ownership stake for investors, denoted by α^{fb} , which then defines the socially optimal level of entrepreneur entry, denoted by n_E^{fb} .

We first derive the total expected utility of all investors, denoted by TU_I . Recall that θ is uniformly distributed with support $[0, \mu_I]$. Thus,

$$\begin{aligned} TU_I &= \tilde{n} \left[\int_0^{\hat{\theta}} \left(\alpha \frac{\delta \rho y}{\phi} \tilde{w} - \theta \tilde{w} \right) \frac{1}{\mu_I} d\theta + \int_{\hat{\theta}}^{\mu_I} \tilde{w} \frac{1}{\mu_I} d\theta \right] \\ &= \tilde{n} \tilde{w} \frac{1}{\mu_I} \left[\left(\alpha \frac{\delta \rho y}{\phi} - 1 \right) \hat{\theta} - \frac{1}{2} (\hat{\theta})^2 + \mu_I \right]. \end{aligned}$$

Using $\hat{\theta} = \alpha \frac{\delta \rho y}{\phi} - 1$ we get

$$TU_I = \tilde{n} \tilde{w} \left[1 + \frac{1}{2\mu_I} \left(\alpha \frac{\delta \rho y}{\phi} - 1 \right)^2 \right].$$

Next consider the total expected utility of all entrepreneurs prior to market entry, denoted by TU_E . Recall that l is uniformly distributed with support $[0, \mu_E]$, and note that the entry cost for the marginal entrepreneur is given by $\hat{l} = U_E = \delta \rho (1 - \alpha) y$. Thus,

$$\begin{aligned} TU_E &= n_E \int_0^{\hat{l}} (\delta \rho (1 - \alpha) y - l) d\Gamma_E (l|L \leq \hat{l}) \\ &= n_E \int_0^{\hat{l}} (\delta \rho (1 - \alpha) y - l) \frac{1}{\hat{l}} dl = \frac{1}{2} n_E \delta \rho (1 - \alpha) y. \end{aligned}$$

Using the entry condition $n_E = \frac{1}{\mu_E} \delta \rho (1 - \alpha) y$ we get $TU_E = \frac{1}{2\mu_E} [\delta \rho (1 - \alpha) y]^2$. Thus, the total welfare W is given by

$$W = TU_E + TU_I = \frac{1}{2\mu_E} [\delta \rho (1 - \alpha) y]^2 + \tilde{n} \tilde{w} \left[1 + \frac{1}{2\mu_I} \left(\alpha \frac{\delta \rho y}{\phi} - 1 \right)^2 \right].$$

The first best equity share for investors, α^{fb} , is then defined by the first-order condition:

$$\frac{1}{\mu_E} \phi \delta \rho (1 - \alpha) y = \tilde{n} \tilde{w} \frac{1}{\mu_I} \left(\alpha \frac{\delta \rho y}{\phi} - 1 \right).$$

Note that this condition, which defines α^{fb} , is identical to the market clearing condition (A.1). Thus, the equilibrium ownership stake for investors is socially efficient, i.e., $\alpha^* = \alpha^{fb}$. The entry condition for entrepreneurs, $n_E = \frac{1}{\mu_E} \delta \rho (1 - \alpha) y$, then implies that the equilibrium number of new ventures is also socially efficient, i.e., $n_E^* = n_E^{fb}$.

Now consider the effect of a founding subsidy S_E . Using uniform distributions for l and θ , the new market equilibrium is defined by the entry condition $n_E = \frac{1}{\mu_E} [\delta \rho (1 - \alpha) y + S_E]$, and the market clearing condition

$$\frac{1}{\mu_E} [\delta \rho (1 - \alpha) y + S_E] \phi = \frac{1}{\mu_I} \left[\alpha \frac{\delta \rho y}{\phi} - 1 \right] \tilde{n} \tilde{w}. \quad (\text{A.2})$$

Using (A.2) we get

$$\frac{d\alpha^*(S_E)}{dS_E} = \frac{\frac{1}{\mu_E} \phi}{\frac{1}{\mu_E} \delta \rho y \phi + \frac{1}{\mu_I} \frac{\delta \rho y}{\phi} \tilde{n} \tilde{w}} > 0.$$

Moreover, using the entry condition,

$$\frac{dn_E^*(S_E)}{dS_E} = \frac{1}{\mu_E} \left[1 - \delta \rho y \frac{d\alpha^*}{dS_E} \right] = \frac{\frac{1}{\mu_E} \frac{1}{\mu_I} \frac{1}{\phi} \tilde{n} \tilde{w}}{\frac{1}{\mu_E} \phi + \frac{1}{\mu_I} \frac{1}{\phi} \tilde{n} \tilde{w}} > 0.$$

Next, consider the effect of a funding subsidy $S_I = \phi s_I$. With $S_I > 0$ the entry condition is still given by $n_E = \frac{1}{\mu_E} \delta \rho (1 - \alpha) y$, while the market clearing condition changes to

$$\frac{1}{\mu_E} \delta \rho (1 - \alpha) y \phi = \frac{1}{\mu_I} \left[\alpha \frac{\delta \rho y}{\phi} - 1 + \frac{1}{\phi} S_I \right] \tilde{n} \tilde{w}. \quad (\text{A.3})$$

Implicitly differentiating (A.3) yields

$$\frac{d\alpha^*(S_I)}{dS_I} = -\frac{\frac{1}{\mu_I} \frac{1}{\phi} \tilde{n} \tilde{w}}{\frac{1}{\mu_E} \delta \rho y \phi + \frac{1}{\mu_I} \frac{\delta \rho y}{\phi} \tilde{n} \tilde{w}} < 0 \quad \frac{dn_E^*(S_I)}{dS_I} = -\frac{1}{\mu_E} \delta \rho y \frac{d\alpha^*}{dS_I} = \frac{\frac{1}{\mu_E} \frac{1}{\mu_I} \frac{1}{\phi} \tilde{n} \tilde{w}}{\frac{1}{\mu_E} \phi + \frac{1}{\mu_I} \frac{1}{\phi} \tilde{n} \tilde{w}} > 0.$$

Finally note that $n_E^*(S_E = 0) = n_E^*(S_I = 0) = n_E^*$ and $dn_E^*(S_E)/dS_E = dn_E^*(S_I)/dS_I$. Thus, $n_E^*(S_E) = n_E^*(S_I) > n_E^*$ for all $S_E = S_I$. Moreover, note that $\alpha^*(S_E = 0) = \alpha^*(S_I = 0) = \alpha^*$. The fact that $d\alpha^*(S_E)/dS_E > 0$ and $d\alpha^*(S_I)/dS_I < 0$ then implies that $\alpha^*(S_E) > \alpha^* > \alpha^*(S_I)$ for all $S_E = S_I$. \square

4 Tax Incidence Equivalent Result

We now formally prove that the tax incidence result also holds in our model. Suppose the government either pays the monetary subsidy S_{I-E} to entrepreneurs, or the subsidy S_I to investors, with $S_{I-E} = S_I$. When offering S_{I-E} to entrepreneurs, the market equilibrium is defined by the entry condition $n_E = \frac{1}{\mu_E} [\delta\rho(1 - \alpha)y + S_{I-E}]$, and the market clearing condition

$$\frac{1}{\mu_E} [\delta\rho(1 - \alpha)y + S_{I-E}] \phi = \frac{1}{\mu_I} \left[\alpha \frac{\delta\rho y}{\phi} - 1 \right] \tilde{n} \tilde{w}. \quad (\text{A.4})$$

Implicitly differentiating (A.4) yields

$$\frac{d\alpha^*(S_{I-E})}{dS_{I-E}} = \frac{\frac{1}{\mu_E} \phi}{\frac{1}{\mu_E} \delta\rho y \phi + \frac{1}{\mu_I} \frac{\delta\rho y}{\phi} \tilde{n} \tilde{w}} > 0.$$

Moreover, using the entry condition and the expression for $d\alpha^*(S_{I-E})/dS_{I-E}$, we get

$$\frac{dn_E^*(S_{I-E})}{dS_{I-E}} = \frac{1}{\mu_E} \left[1 - \delta\rho y \frac{d\alpha^*(S_{I-E})}{dS_{I-E}} \right] = \frac{\frac{1}{\mu_E} \frac{1}{\mu_I} \frac{1}{\phi} \tilde{n} \tilde{w}}{\frac{1}{\mu_E} \phi + \frac{1}{\mu_I} \frac{1}{\phi} \tilde{n} \tilde{w}}.$$

Finally, using the expression for $dn_E^*(S_I)/dS_I$ as derived in Proof of Proposition 1, we can immediately see that $dn_E^*(S_{I-E})/dS_{I-E} = dn_E^*(S_I)/dS_I$. Thus, $n_E^*(S_{I-E}) = n_E^*(S_I)$ for all $S_{I-E} = S_I$. Consequently it does not matter in our benchmark model whether entrepreneurs or investors get the monetary subsidy; it always leads to the same equilibrium level of entrepreneurship n_E^* .

5 Model with Intergenerational Dynamics: Equilibria

Using that θ is uniformly distributed with support $[0, \mu_I]$, we can write the expected utility of an entrepreneur in period t as

$$\begin{aligned} U_{E,t} &= \delta\rho(1 - \alpha_t)y \left[\int_0^{\hat{\theta}_{t+1}} \left(\alpha_{t+1} \frac{\delta\rho y}{\phi} - \theta \right) \frac{1}{\mu_I} d\theta + \int_{\hat{\theta}_{t+1}}^{\mu_I} \frac{1}{\mu_I} d\theta \right] \\ &= \delta\rho(1 - \alpha_t)y \frac{1}{\mu_I} \left[\left(\alpha_{t+1} \frac{\delta\rho y}{\phi} - 1 \right) \hat{\theta}_{t+1} - \frac{1}{2} \hat{\theta}_{t+1}^2 + \mu_I \right]. \end{aligned}$$

Using $\widehat{\theta}_{t+1} = \alpha_{t+1} \frac{\delta \rho y}{\phi} - 1$ we get

$$U_{E,t} = \delta \rho (1 - \alpha_t) y \left[1 + \frac{1}{2\mu_I} \left(\alpha_{t+1} \frac{\delta \rho y}{\phi} - 1 \right)^2 \right].$$

The market equilibrium $\{n_{E,t}^*(n_{E,t-1}), \alpha_t^*(n_{E,t-1})\}$ in period t is then defined by (i) the entry condition for entrepreneurs, $n_{E,t} = \frac{1}{\mu_E} U_{E,t}$, and the market clearing condition

$$n_{E,t} \phi = \frac{1}{\mu_I} \left[\alpha_t \frac{\delta \rho y}{\phi} - 1 \right] \rho n_{E,t-1} (1 - \alpha_{t-1}) y.$$

Using the equilibrium conditions we define

$$J \equiv \frac{1}{\mu_E} \delta \rho (1 - \alpha_t) y \left[1 + \frac{1}{2\mu_I} \left(\alpha_{t+1} \frac{\delta \rho y}{\phi} - 1 \right)^2 \right] - n_{E,t} = 0$$

$$H \equiv \frac{1}{\mu_I} \left(\alpha_t \frac{\delta \rho y}{\phi} - 1 \right) \rho n_{E,t-1} (1 - \alpha_{t-1}) y - n_{E,t} \phi = 0.$$

Using Cramer's rule we get

$$\frac{dn_{E,t}^*(n_{E,t-1})}{dn_{E,t-1}} = \frac{\begin{vmatrix} -\frac{\partial J}{\partial n_{E,t-1}} & \frac{\partial J}{\partial \alpha_t} \\ -\frac{\partial H}{\partial n_{E,t-1}} & \frac{\partial H}{\partial \alpha_t} \end{vmatrix}}{\begin{vmatrix} \frac{\partial J}{\partial n_{E,t}} & \frac{\partial J}{\partial \alpha_t} \\ \frac{\partial H}{\partial n_{E,t}} & \frac{\partial H}{\partial \alpha_t} \end{vmatrix}} = \frac{-\frac{\partial J}{\partial n_{E,t-1}} \frac{\partial H}{\partial \alpha_t} + \frac{\partial H}{\partial n_{E,t-1}} \frac{\partial J}{\partial \alpha_t}}{\frac{\partial J}{\partial n_{E,t}} \frac{\partial H}{\partial \alpha_t} - \frac{\partial H}{\partial n_{E,t}} \frac{\partial J}{\partial \alpha_t}},$$

where $\partial J / \partial n_{E,t-1} = 0$, $\partial J / \partial n_{E,t} = -1$, $\partial H / \partial n_{E,t} = -\phi$, and

$$\frac{\partial H}{\partial n_{E,t-1}} = \frac{1}{\mu_I} \left(\alpha_t \frac{\delta \rho y}{\phi} - 1 \right) \rho (1 - \alpha_{t-1}) y > 0 \quad \frac{\partial H}{\partial \alpha_t} = \frac{1}{\mu_I} \frac{\delta \rho y}{\phi} \rho n_{E,t-1} (1 - \alpha_{t-1}) y > 0$$

$$\frac{\partial J}{\partial \alpha_t} = -\frac{1}{\mu_E} \delta \rho y \left[1 + \frac{1}{2\mu_I} \left(\alpha_{t+1} \frac{\delta \rho y}{\phi} - 1 \right)^2 \right] < 0.$$

For $\alpha_t^* > \phi/(\delta\rho y)$ we then get

$$\frac{dn_{E,t}^*(n_{E,t-1})}{dn_{E,t-1}} = \frac{\overbrace{\frac{\partial H}{\partial n_{E,t-1}}}^{>0} \overbrace{\frac{\partial J}{\partial \alpha_t}}^{<0}}{-\underbrace{\frac{\partial H}{\partial \alpha_t}}_{>0} + \phi \underbrace{\frac{\partial J}{\partial \alpha_t}}_{<0}} > 0.$$

Moreover, note that there is no capital supply when $n_{E,t-1} = 0$. Thus, $n_{E,t}^*(0) = 0$.

Next we identify and characterize the steady state equilibria. In the steady state we have $n_{E,t} = n_{E,t-1}$ and $\alpha_t = \alpha_{t-1}$. Adjusting the market clearing condition for the steady state with $n_E \equiv n_{E,t} = n_{E,t-1}$ and $\alpha \equiv \alpha_t = \alpha_{t-1}$, we define

$$\underbrace{n_E \phi}_{\equiv E(\alpha)} = \frac{1}{\mu_I} \underbrace{\left(\alpha \frac{\delta\rho y}{\phi} - 1 \right) \rho n_E (1 - \alpha) y}_{\equiv I(\alpha)}, \quad (\text{A.5})$$

where $E(\alpha)$ is the total capital demand, and $I(\alpha)$ is the total capital supply. Let $\Psi(\alpha) \equiv I(\alpha)/E(\alpha)$ denote the excess supply function. The steady state market clearing condition (A.5) implies that

$$\Psi(\alpha) = \frac{1}{\phi} \frac{1}{\mu_I} \left(\alpha \frac{\delta\rho y}{\phi} - 1 \right) \rho (1 - \alpha) y = 1$$

in the steady state equilibrium, where $\alpha \frac{\delta\rho y}{\phi} - 1 = \hat{\theta} \geq 0$. Next we analyze the shape of $\Psi(\alpha)$. We get

$$\frac{d\Psi(\alpha)}{d\alpha} = \frac{1}{\phi} \frac{1}{\mu_I} \rho y \left[\underbrace{\frac{\delta\rho y}{\phi} (1 - \alpha)}_{\equiv Z_1} - \underbrace{\left(\alpha \frac{\delta\rho y}{\phi} - 1 \right)}_{\equiv Z_2} \right].$$

Clearly, Z_1 is positive and decreasing in α , while $Z_2 = \hat{\theta}$ is also positive but decreasing in α . Note that $Z_2 = \hat{\theta} = \alpha \frac{\delta\rho y}{\phi} - 1 = 0$ for $\alpha \leq \phi/(\delta\rho y)$. Thus, $\Psi(\alpha) = 0$ for all $\alpha \leq \phi/(\delta\rho y)$. Moreover, $\Psi(1) = 0$. This implies that $\Psi(\alpha)$ has an inverted U-shape, with $\Psi(\alpha) = 0$ for all $\alpha \leq \phi/(\delta\rho y)$ and $\Psi(1) = 0$. This also implies that there exists a unique α , denoted by $\bar{\alpha}$, which maximizes $\Psi(\alpha)$. Using the first-order condition we find that $\bar{\alpha} = \left(1 + \frac{\delta\rho y}{\phi} \right) / \left(\frac{2\delta\rho y}{\phi} \right)$. Moreover, the second-order condition confirms that we have a maximum at $\alpha = \bar{\alpha}$. Evaluating $\Psi(\alpha)$ at $\alpha = \bar{\alpha}$ yields

$$\Psi(\bar{\alpha}) = \frac{1}{4\delta} \frac{1}{\mu_I} \left(\frac{\delta\rho y}{\phi} - 1 \right)^2.$$

Note that $\partial\Psi(\bar{\alpha})/\partial\phi < 0$, with $\lim_{\phi \rightarrow 0} \Psi(\bar{\alpha}) = \infty$ and $\Psi(\bar{\alpha}) = 0$ for all $\phi \geq \delta\rho y$.¹ Thus, there exists a threshold $\hat{\phi} \in (0, \delta\rho y)$ such that

- (i) for $\phi > \hat{\phi}$ there exists no value of α which satisfies $\Psi(\alpha) = 1$,
- (ii) for $\phi = \hat{\phi}$ there exists a unique α , namely $\bar{\alpha}$, so that $\Psi(\alpha = \bar{\alpha}) = 1$, and
- (iii) for $\phi < \hat{\phi}$ there exist two values of α , denoted by α' and α'' , with $\alpha' < \alpha''$, which both satisfy $\Psi(\alpha) = 1$.

Moreover, given the inverted U-shape of $\Psi(\alpha)$ we can infer that

$$\left. \frac{d\Psi(\alpha)}{d\alpha} \right|_{\alpha=\alpha'} > 0 \quad \text{and} \quad \left. \frac{d\Psi(\alpha)}{d\alpha} \right|_{\alpha=\alpha''} < 0.$$

We can now characterize the steady state equilibria in terms of entrepreneur entry $n_{E,t}$. Recall that the market is competitive, so when entering the market entrepreneurs take their future equity share α_{t+1} as given. From the entry condition we can see that α_t uniquely defines the equilibrium number of entrepreneurs, $n_{E,t}$. Moreover, $dn_{E,t}/d\alpha_t < 0$. We already know that $n_{E,t} = n_{E,t-1} = 0$ always constitutes a steady state equilibrium. Moreover, for $\phi < \hat{\phi}$ we know that there exist two values of α , α' and α'' , which satisfy the steady state market clearing condition (A.5). For $\phi < \hat{\phi}$ we thus get two additional steady state equilibria, which we define as $n_E^M(\alpha^M)$ and $n_E^H(\alpha^H)$, where ‘H’ stands for ‘high’ and ‘M’ stands for ‘medium’ (where, using our original notation, $\alpha^M = \alpha''$ and $\alpha^H = \alpha'$). For $\phi = \hat{\phi}$ we know that the steady state market clearing condition (A.5) is satisfied for $\alpha = \bar{\alpha}$. Consequently, for $\phi = \hat{\phi}$ we have only one additional steady state equilibrium (in addition to $n_{E,t} = n_{E,t-1} = 0$): $n_E^M(\alpha^M)$.

It remains to verify which steady state equilibria are stable vs. unstable. A steady state equilibrium is stable if

$$\left. \frac{dn_{E,t}^*(n_{E,t-1})}{dn_{E,t-1}} \right|_{\substack{n_{E,t}=n_{E,t-1} \\ \alpha_t=\alpha_{t-1}}} < 1.$$

Using the partial derivatives we get

$$\frac{dn_{E,t}^*(n_{E,t-1})}{dn_{E,t-1}} = \frac{\frac{1}{\mu_I} \left(\alpha_t \frac{\delta\rho y}{\phi} - 1 \right) \rho (1 - \alpha_{t-1}) y \frac{1}{\mu_E} \left[1 + \frac{1}{2\mu_I} \left(\alpha_{t+1} \frac{\delta\rho y}{\phi} - 1 \right)^2 \right]}{\frac{1}{\mu_I} \frac{1}{\phi} \rho n_{E,t-1} (1 - \alpha_{t-1}) y + \phi \frac{1}{\mu_E} \left[1 + \frac{1}{2\mu_I} \left(\alpha_{t+1} \frac{\delta\rho y}{\phi} - 1 \right)^2 \right]}.$$

¹Note that the expected net payoff from a venture is negative when $\phi > \delta\rho y$. In this case it can never be optimal for investors to invest ϕ .

First consider the zero steady state equilibrium $n_{E,t} = n_{E,t-1} = 0$. Note that $n_{E,t-1} = 0$ corresponds to $\alpha_t = \phi/(\delta\rho y)$, in which case there is no capital supply (and therefore $n_{E,t} = 0$). Evaluating the derivative at $n_{E,t} = n_{E,t-1} = 0$ and $\alpha_t = \alpha_{t-1} = \phi/(\delta\rho y)$ we get

$$\left. \frac{dn_E^*}{dn_{E-1}} \right|_{\substack{n_{E,t}=n_{E,t-1}=0 \\ \alpha_t=\alpha_{t-1}=\phi/(\delta\rho y)}} = 0.$$

Thus, the zero steady state equilibrium $n_{E,t} = n_{E,t-1} = 0$ is stable. Next consider the two additional steady state equilibria for $\phi < \hat{\phi}$. For the high steady state equilibrium $n_{E,t}^H(\alpha^H) = n_{E,t-1}^H(\alpha^H)$ we get

$$\left. \frac{dn_E^*}{dn_{E-1}} \right|_{\substack{n_{E,t}=n_{E,t-1}=n_{E,t}^H \\ \alpha_t=\alpha_{t-1}=\alpha^H}} = \frac{\frac{1}{\mu_I} \left(\alpha^H \frac{\delta\rho y}{\phi} - 1 \right) \rho(1 - \alpha^H) y \frac{1}{\mu_E} \left[1 + \frac{1}{2\mu_I} \left(\alpha_{t+1} \frac{\delta\rho y}{\phi} - 1 \right)^2 \right]}{\frac{1}{\mu_I} \frac{1}{\phi} \rho n_{E,t}^H (1 - \alpha^H) y + \phi \frac{1}{\mu_E} \left[1 + \frac{1}{2\mu_I} \left(\alpha_{t+1} \frac{\delta\rho y}{\phi} - 1 \right)^2 \right]}.$$

This is smaller than one if

$$\frac{1}{\mu_I} \left(\alpha^H \frac{\delta\rho y}{\phi} - 1 \right) \rho(1 - \alpha^H) y \frac{1}{\mu_E} Z < \frac{1}{\mu_I} \frac{1}{\phi} \rho n_{E,t}^H (1 - \alpha^H) y + \phi \frac{1}{\mu_E} Z, \quad (\text{A.6})$$

where

$$Z = 1 + \frac{1}{2\mu_I} \left(\alpha_{t+1} \frac{\delta\rho y}{\phi} - 1 \right)^2.$$

Using the corresponding entry condition $n_{E,t}^H = \frac{1}{\mu_E} \delta\rho (1 - \alpha^H) y Z$ to replace $n_{E,t}^H$ in (A.6) we get

$$\frac{1}{\mu_I} \left(\alpha^H \frac{\delta\rho y}{\phi} - 1 \right) \rho(1 - \alpha^H) y < \frac{1}{\mu_I} \frac{1}{\phi} \delta [\rho (1 - \alpha^H) y]^2 + \phi. \quad (\text{A.7})$$

We can then use (A.5) to replace ϕ in (A.7) and get

$$0 < \frac{1}{\mu_I} \frac{1}{\phi} \delta [\rho (1 - \alpha^H) y]^2.$$

This condition is clearly satisfied as $\alpha^H < 1$. Thus, the high steady state equilibrium $n_{E,t}^H(\alpha^H) = n_{E,t-1}^H(\alpha^H)$ is stable. Finally recall that $dn_{E,t}^*/dn_{E,t-1} \geq 0$, with

$$\left. \frac{dn_E^*}{dn_{E-1}} \right|_{\substack{n_{E,t}=n_{E,t-1}=0 \\ \alpha_t=\alpha_{t-1}=\phi/(\delta\rho y)}} , \quad \left. \frac{dn_E^*}{dn_{E-1}} \right|_{\substack{n_{E,t}=n_{E,t-1}=n_{E,t}^H \\ \alpha_t=\alpha_{t-1}=\alpha^H}} < 1.$$

This implies that

$$\left. \frac{dn_E^*}{dn_{E-1}} \right|_{\substack{n_{E,t}=n_{E,t-1}=n_E^M \\ \alpha_t=\alpha_{t-1}=\alpha^M}} > 1.$$

Thus, the medium steady state equilibrium $n_{E,t}^M(\alpha^M) = n_{E,t-1}^M(\alpha^M)$ is unstable.

6 Proof of Proposition 2

We first derive the total expected utility of all investors in the high steady state equilibrium, denoted by TU_I . Using $\hat{\theta} = \alpha \frac{\delta \rho y}{\phi} - 1$ we get

$$TU_I = \rho n_E w \left[\int_0^{\hat{\theta}} \left(\alpha \frac{\delta \rho y}{\phi} - \theta \right) \frac{1}{\mu_I} d\theta + \int_{\hat{\theta}}^{\mu_I} \frac{1}{\mu_I} d\theta \right] = \rho n_E w \left[1 + \frac{1}{2\mu_I} \left(\alpha \frac{\delta \rho y}{\phi} - 1 \right)^2 \right],$$

where ρn_E is the number of investors in each period, and $w = (1 - \alpha)y$ is the equilibrium wealth of investors in the steady state.

Next, recall from Proof of Proposition 1 that the total expected utility of all entrepreneurs prior to entry, TU_E , is given by $TU_E = \frac{1}{2} n_E \delta \rho (1 - \alpha) y$. Using the entry condition for the high steady state equilibrium,

$$n_E = \frac{1}{\mu_E} U_E = \frac{1}{\mu_E} \delta \rho (1 - \alpha) y \left[1 + \frac{1}{2\mu_I} \left(\alpha_{t+1} \frac{\delta \rho y}{\phi} - 1 \right)^2 \right], \quad (\text{A.8})$$

we can write the total steady state welfare function $W = \frac{1}{1-\delta} (TU_E + TU_I)$ as

$$W = \underbrace{\frac{1}{1-\delta} \frac{1}{\mu_E} \delta (\rho y)^2}_{\equiv X > 0} Z_{t+1} (1 - \alpha)^2 \left[\frac{1}{2} \delta + \left[1 + \frac{1}{2\mu_I} \left(\alpha \frac{\delta \rho y}{\phi} - 1 \right)^2 \right] \right],$$

where

$$Z_{t+1} = 1 + \frac{1}{2\mu_I} \left(\alpha_{t+1} \frac{\delta \rho y}{\phi} - 1 \right)^2.$$

To show that the equilibrium equity stake α^* does not maximize W , we derive $dW/d\alpha$, and then evaluate the derivative at $\alpha = \alpha^*$. We get

$$\frac{dW}{d\alpha} = X \left[-2(1 - \alpha) \left[\frac{1}{2} \delta + \left[1 + \frac{1}{2\mu_I} \left(\alpha \frac{\delta \rho y}{\phi} - 1 \right)^2 \right] \right] + (1 - \alpha)^2 \frac{1}{\mu_I} \left(\alpha \frac{\delta \rho y}{\phi} - 1 \right) \frac{\delta \rho y}{\phi} \right].$$

The market clearing condition (A.5) for the high steady state equilibrium, which defines α^* , can be written as

$$(1 - \alpha) = \frac{\phi}{\frac{1}{\mu_I} \left[\alpha \frac{\delta \rho y}{\phi} - 1 \right] \rho y}.$$

Using this expression we can evaluate $dW/d\alpha$ at $\alpha = \alpha^*$, and get after simplifying

$$\left. \frac{dW}{dV} \right|_{\alpha=\alpha^*} = -2X \frac{\phi}{\frac{1}{\mu_I} \left[\alpha \frac{\delta \rho y}{\phi} - 1 \right] \rho y} \left[1 + \frac{1}{2\mu_I} \left(\alpha \frac{\delta \rho y}{\phi} - 1 \right)^2 \right] < 0.$$

This implies that $\alpha^* > \alpha^{fb}$. Moreover, we can see from (A.8) that $dn_E^*/d\alpha < 0$. Thus, $n_E^* < n_E^{fb}$. \square

7 Proof of Proposition 3

With a founding subsidy S_E , the high steady state equilibrium is defined by the following entry condition and market clearing condition:

$$n_E = \frac{1}{\mu_E} \left[\delta \rho (1 - \alpha) y \left[1 + \frac{1}{2\mu_I} \left(\alpha_{t+1} \frac{\delta \rho y}{\phi} - 1 \right)^2 \right] + S_E \right] \quad (\text{A.9})$$

$$\phi = \frac{1}{\mu_I} \left(\alpha \frac{\delta \rho y}{\phi} - 1 \right) \rho (1 - \alpha) y, \quad (\text{A.10})$$

where (A.10) defines α^* , and (A.9) then defines n_E^* . From (A.10) we can immediately see that $d\alpha^*(S_E)/dS_E = 0$. Moreover, using (A.9) we find

$$\frac{dn_E^*(S_E)}{dS_E} = \frac{1}{\mu_E} \left[-\delta \rho y \underbrace{\frac{d\alpha^*(S_E)}{dS_E}}_{=0} \left[1 + \frac{1}{2\mu_I} \left(\alpha_{t+1} \frac{\delta \rho y}{\phi} - 1 \right)^2 \right] + 1 \right] = \frac{1}{\mu_E} > 0.$$

\square

8 Proof of Proposition 4

Recall that θ is uniformly distributed with support $[0, \mu_I]$, and $\hat{\theta}_{t+1} = \alpha_{t+1} \frac{\delta \rho y}{\phi} - 1$. With a funding subsidy S_I , the expected utility of an entrepreneur in the high steady state equilibrium, $U_E(S_I)$, is then given by

$$\begin{aligned} U_E(S_I) &= \rho \delta (1 - \alpha) y \left[\int_0^{\hat{\theta}_{t+1} + \frac{1}{\phi} S_I} \left(\alpha_{t+1} \frac{\delta \rho y}{\phi} + \frac{1}{\phi} S_I - \theta \right) \frac{1}{\mu_I} d\theta + \int_{\hat{\theta}_{t+1} + \frac{1}{\phi} S_I}^{\mu_I} \frac{1}{\mu_I} d\theta \right] \\ &= \rho \delta (1 - \alpha) y \left[1 + \frac{1}{2\mu_I} \left(\alpha_{t+1} \frac{\delta \rho y}{\phi} - 1 + \frac{1}{\phi} S_I \right)^2 \right]. \end{aligned}$$

Thus, with a funding subsidy S_I , the entry condition and the market clearing condition become

$$n_E = \frac{1}{\mu_E} \rho \delta (1 - \alpha) y \left[1 + \frac{1}{2\mu_I} \left(\alpha_{t+1} \frac{\delta \rho y}{\phi} - 1 + \frac{1}{\phi} S_I \right)^2 \right] \quad (\text{A.11})$$

$$\phi = \frac{1}{\mu_I} \left(\alpha \frac{\delta \rho y}{\phi} - 1 + \frac{1}{\phi} S_I \right) \rho (1 - \alpha) y, \quad (\text{A.12})$$

where (A.12) defines α^* , and (A.11) then defines n_E^* . Using (A.12) we define

$$H \equiv \frac{1}{\mu_I} \left(\alpha \frac{\delta \rho y}{\phi} - 1 + \frac{1}{\phi} S_I \right) \rho (1 - \alpha) y - \phi = 0.$$

Using H we get

$$\frac{d\alpha^*(S_I)}{dS_I} = - \frac{\frac{1}{\mu_I} \frac{1}{\phi} \rho (1 - \alpha) y}{\frac{\partial H}{\partial \alpha}}.$$

Recall the excess supply function $\Psi(\alpha)$ from our formal characterization of the dynamic equilibrium. With a funding subsidy S_I the excess supply function can be written as

$$\Psi(\alpha, S_I) = \frac{1}{\phi} \frac{1}{\mu_I} \left(\delta \rho \frac{1}{\phi} \alpha y - 1 + \frac{1}{\phi} S_I \right) \rho (1 - \alpha) y.$$

Furthermore, for the high steady state equilibrium with $\alpha = \alpha^H$ we know that $\left. \frac{\partial \Psi(\alpha, S_I)}{\partial \alpha} \right|_{\alpha = \alpha^H} > 0$. Clearly, $H = \phi \Psi(\alpha, S_I) - \phi$. Hence, $\partial H / \partial \alpha > 0$, so that $d\alpha^*(S_I) / dS_I < 0$.

Finally, using the entry condition (A.11) we get

$$\frac{dn_E^*(S_I)}{dS_I} = \frac{1}{\mu_E} \rho \delta y \left[\underbrace{\frac{d\alpha^*(S_I)}{dS_I}}_{<0} \left[1 + \frac{1}{2\mu_I} Z_{t+1}^2 \right] + (1 - \alpha) \frac{1}{\mu_I} Z_{t+1} \frac{1}{\phi} \right],$$

where

$$Z_{t+1} = \alpha_{t+1} \frac{\delta \rho y}{\phi} - 1 + \frac{1}{\phi} S_I > 0.$$

Thus, $dn_E^*(S_I)/dS_I > 0$. □

9 Proof of Proposition 5

Note that $n_E^*(S_I = 0) = n_E^*(S_E = 0) = n_E^*$. Thus, to show that $n_E^*(S_I) > n_E^*(S_E)$ for all $S_I = S_E$ it is sufficient to show that $dn_E^*(S_I)/dS_I > dn_E^*(S_E)/dS_E$ for all $S_I = S_E$. Recall from Proof of Proposition 3 that $dn_E^*(S_E)/dS_E = 1/\mu_E$. Moreover, using the derivations from Proof of Proposition 4 we get

$$\frac{dn_E^*(S_I)}{dS_I} = \frac{1}{\mu_E} \rho \delta y \left[\frac{\frac{1}{\mu_I} \frac{1}{\phi} \rho (1 - \alpha) y}{\frac{\partial H}{\partial \alpha}} \left[1 + \frac{1}{2\mu_I} Z_{t+1}^2 \right] + (1 - \alpha) \frac{1}{\mu_I} Z_{t+1} \frac{1}{\phi} \right],$$

where

$$Z_{t+1} = \alpha_{t+1} \frac{\delta \rho y}{\phi} - 1 + \frac{1}{\phi} S_I > 0,$$

and $\partial H/\partial \alpha > 0$ (see Proof of Proposition 4). Note that

$$\frac{\partial H}{\partial \alpha} = \frac{1}{\mu_I} \rho y \left[\frac{\delta \rho y}{\phi} (1 - \alpha) - \left(\alpha \frac{\delta \rho y}{\phi} - 1 + \frac{1}{\phi} S_I \right) \right] > 0. \quad (\text{A.13})$$

We then get after simplifying

$$\frac{dn_E^*(S_I)}{dS_I} = \frac{\frac{1}{\mu_E} \rho \delta y (1 - \alpha) \left[1 + \frac{1}{2\mu_I} Z_{t+1}^2 + \frac{\delta \rho y}{\phi} (1 - \alpha) \frac{1}{\mu_I} Z_{t+1} - \frac{1}{\mu_I} Z Z_{t+1} \right]}{\delta \rho y (1 - \alpha) - \phi Z},$$

where

$$Z = \alpha \frac{\delta \rho y}{\phi} - 1 + \frac{1}{\phi} S_I > 0.$$

Thus, $dn_E^*(S_I)/dS_I > dn_E^*(S_E)/dS_E$ if

$$\frac{\frac{1}{\mu_E}\delta\rho y(1-\alpha)\left[1+\frac{1}{2\mu_I}Z_{t+1}^2+\frac{\delta\rho y}{\phi}(1-\alpha)\frac{1}{\mu_I}Z_{t+1}-\frac{1}{\mu_I}ZZ_{t+1}\right]}{\delta\rho y(1-\alpha)-\phi Z} > \frac{1}{\mu_E},$$

which can be rearranged to

$$\delta\rho y(1-\alpha)Z_{t+1}\frac{1}{\mu_I}\left[\frac{1}{2}Z_{t+1}+\underbrace{\frac{\delta\rho y}{\phi}(1-\alpha)-Z}_{\equiv X}\right] > -\phi Z. \quad (\text{A.14})$$

We can immediately infer from (A.13) that $X > 0$, so that (A.14) is satisfied. Consequently, $n_E^*(S_I) > n_E^*(S_E)$ for all $S_I = S_E$.

Finally note that $\alpha^*(S_I = 0) = \alpha^*(S_E = 0) = \alpha^*$. Moreover, we know from Proposition 3 that $d\alpha^*(S_E)/dS_E = 0$, and from Proposition 4 that $d\alpha^*(S_I)/dS_I < 0$. Thus, $\alpha^*(S_I) < \alpha^*(S_E) = \alpha^*$ for all $S_I = S_E$. \square

10 Serial Entrepreneurs and Angels

We now show that our main insight, namely that funding subsidies generate more entrepreneurial activities than founding subsidies in the presence of intergenerational linkages, remains intact when allowing for serial entrepreneurs and angels. For simplicity we consider the dynamic model without external angels ($\tilde{n} = 0$) and focus on the high steady state equilibrium.

Suppose that each entrepreneur can start another venture in the next period with probability σ_E , and that each investor can make another investment in the next period with probability σ_I . For tractability we assume that formerly successful entrepreneurs can start a new venture and make angel investments at the same time. In this case we also assume that the entire wealth is invested in other start-ups or in the safe asset, so that ventures started by wealthy serial entrepreneurs are financed by different angels.

We first characterize the steady state equilibrium. Using the uniform distribution for θ and $\hat{\theta} = \alpha\frac{\delta\rho y}{\phi} - 1$, we can write the marginal return for an investor, denoted R , as

$$R = \int_0^{\hat{\theta}} \left(\alpha\frac{\delta\rho y}{\phi} - \theta\right) \frac{1}{\mu_I} d\theta + \int_{\hat{\theta}}^{\mu_I} \frac{1}{\mu_I} d\theta = 1 + \frac{1}{2\mu_I} \left(\alpha\frac{\delta\rho y}{\phi} - 1\right)^2.$$

To ensure that investors reinvest their wealth whenever possible (serial angels), we assume that $\delta R > 1$.

Let U_I^S denote the expected utility of a ‘first-time’ serial investor, i.e., an investor who just succeeded as entrepreneur. We assume that σ_I is sufficiently small so that $\sigma_I \delta R < 1$. In the steady state U_I^S is then given by

$$U_I^S = (1 - \sigma_I) w_0 R [1 + \sigma_I \delta R + (\sigma_I \delta R)^2 + \dots] = (1 - \sigma_I) w_0 \frac{R}{1 - \sigma_I \delta R},$$

where $w_0 = (1 - \alpha)y$ is the investor’s initial wealth. The expected utility of a serial entrepreneur, denoted by U_E^S , is then given by

$$U_E^S = \delta \rho U_{I,t+1}^S + \sigma_E \delta U_E^S, \quad (\text{A.15})$$

where

$$U_{I,t+1}^S = (1 - \sigma_I) w_0 \frac{R_{t+1}}{1 - \sigma_I \delta R_{t+1}}, \quad R_{t+1} = 1 + \frac{1}{2\mu_I} \left(\alpha_{t+1} \frac{\delta \rho y}{\phi} - 1 \right)^2.$$

Solving (A.15) for U_E^S we get $U_E^S = \frac{1}{1 - \sigma_E \delta} \delta \rho U_{I,t+1}^S$, so that the entry condition for entrepreneurs can be written as

$$n_E = \frac{1}{\mu_E} U_E^S = \frac{1}{\mu_E} \delta \rho (1 - \alpha) y \frac{1 - \sigma_I}{1 - \sigma_E \delta} \frac{R_{t+1}}{1 - \sigma_I \delta R_{t+1}}.$$

Moreover, the market clearing condition is given by

$$N_E^S \phi = \frac{1}{\mu_I} \left(\alpha \frac{\delta \rho y}{\phi} - 1 \right) K,$$

where N_E^S is the total number of entrepreneurs in the market, and K is the total stock of capital. In the steady state we have

$$N_E^S = n_E^S + \sigma_E n_{E-1}^S + \sigma_E^2 n_{E-2}^S + \dots = n_E^S [1 + \sigma_E + \sigma_E^2 + \dots] = \frac{1}{1 - \sigma_E} n_E^S$$

$$\begin{aligned} K &= \rho n_{E-1}^S w_0 + \sigma_I \rho n_{E-2}^S w_0 R + \sigma_I^2 \rho n_{E-3}^S w_0 R^2 + \dots = \rho n_E^S w_0 [1 + \sigma_I R + (\sigma_I R)^2 + \dots] \\ &= \frac{1}{1 - \sigma_I R} \rho (1 - \alpha) y n_E^S. \end{aligned}$$

Using the definition of R we can then write the market clearing condition as follows:

$$\left(1 - \sigma_I - \sigma_I \frac{1}{2\mu_I} \left(\alpha \frac{\delta\rho y}{\phi} - 1\right)^2\right) \phi = \frac{1}{\mu_I} \left(\alpha \frac{\delta\rho y}{\phi} - 1\right) (1 - \sigma_E) \rho (1 - \alpha) y.$$

Now consider the effect of a founding subsidy S_E . Using the entry condition and market clearing condition we define

$$J \equiv \frac{1}{\mu_E} \left[\delta\rho(1 - \alpha)y \frac{1 - \sigma_I}{1 - \sigma_E\delta} \frac{R_{t+1}}{1 - \sigma_I\delta R_{t+1}} + S_E \right] - n_E = 0$$

$$H \equiv \frac{1}{\mu_I} \left(\alpha \frac{\delta\rho y}{\phi} - 1\right) (1 - \sigma_E) \rho (1 - \alpha) y - \left(1 - \sigma_I - \sigma_I \frac{1}{2\mu_I} \left(\alpha \frac{\delta\rho y}{\phi} - 1\right)^2\right) \phi = 0.$$

We get

$$\frac{dn_E^*(S_E)}{dS_E} = \frac{\begin{vmatrix} -\frac{\partial J}{\partial S_E} & \frac{\partial J}{\partial \alpha} \\ -\frac{\partial H}{\partial S_E} & \frac{\partial H}{\partial \alpha} \end{vmatrix}}{\begin{vmatrix} \frac{\partial J}{\partial n_E} & \frac{\partial J}{\partial \alpha} \\ \frac{\partial H}{\partial n_E} & \frac{\partial H}{\partial \alpha} \end{vmatrix}} = \frac{-\frac{\partial J}{\partial S_E} \frac{\partial H}{\partial \alpha} + \frac{\partial H}{\partial S_E} \frac{\partial J}{\partial \alpha}}{\frac{\partial J}{\partial n_E} \frac{\partial H}{\partial \alpha} - \frac{\partial H}{\partial n_E} \frac{\partial J}{\partial \alpha}},$$

where $\partial J/\partial S_E = 1/\mu_E$, $\partial H/\partial S_E = 0$, $\partial J/\partial n_E = -1$, and $\partial H/\partial n_E = 0$. Thus, $dn_E^*(S_E)/dS_E = \partial J/\partial S_E = 1/\mu_E > 0$.

Next, consider the effect of a funding subsidy S_I . The marginal return function for an investor, R , then becomes

$$R = \int_0^{\hat{\theta} + \frac{1}{\phi} S_I} \left(\alpha \frac{\delta\rho y}{\phi} - \theta + \frac{1}{\phi} S_I\right) \frac{1}{\mu_I} d\theta + \int_{\hat{\theta} + \frac{1}{\phi} S_I}^{\mu_I} \frac{1}{\mu_I} d\theta = 1 + \frac{1}{2\mu_I} \left(\alpha \frac{\delta\rho y}{\phi} - 1 + \frac{1}{\phi} S_I\right)^2.$$

With $S_I > 0$ the high steady state market equilibrium is then defined by the following entry condition and market clearing condition:

$$J \equiv \frac{1}{\mu_E} \delta\rho(1 - \alpha)y \frac{1 - \sigma_I}{1 - \sigma_E\delta} \frac{R_{t+1}}{1 - \sigma_I\delta R_{t+1}} - n_E = 0$$

$$H \equiv \frac{1}{\mu_I} \left(\alpha \frac{\delta\rho y}{\phi} - 1 + \frac{1}{\phi} S_I\right) (1 - \sigma_E) \rho (1 - \alpha) y - \left(1 - \sigma_I - \sigma_I \frac{1}{2\mu_I} \left(\alpha \frac{\delta\rho y}{\phi} - 1 + \frac{1}{\phi} S_I\right)^2\right) \phi = 0,$$

where

$$R_{t+1} = 1 + \frac{1}{2\mu_I} \left(\alpha_{t+1} \frac{\delta\rho y}{\phi} - 1 + \frac{1}{\phi} S_I \right)^2.$$

Using the equilibrium conditions we get

$$\frac{dn_E^*(S_I)}{dS_I} = \frac{\begin{vmatrix} -\frac{\partial J}{\partial S_I} & \frac{\partial J}{\partial \alpha} \\ -\frac{\partial H}{\partial S_I} & \frac{\partial H}{\partial \alpha} \end{vmatrix}}{\begin{vmatrix} \frac{\partial J}{\partial n_E} & \frac{\partial J}{\partial \alpha} \\ \frac{\partial H}{\partial n_E} & \frac{\partial H}{\partial \alpha} \end{vmatrix}} = \frac{-\frac{\partial J}{\partial S_I} \frac{\partial H}{\partial \alpha} + \frac{\partial H}{\partial S_I} \frac{\partial J}{\partial \alpha}}{\frac{\partial J}{\partial n_E} \frac{\partial H}{\partial \alpha} - \frac{\partial H}{\partial n_E} \frac{\partial J}{\partial \alpha}},$$

where $\partial J/\partial n_E = -1$, $\partial H/\partial n_E = 0$, and

$$\begin{aligned} \frac{\partial J}{\partial S_I} &= \frac{1}{\mu_E} \delta\rho(1-\alpha)y \frac{1-\sigma_I}{1-\sigma_E\delta} \frac{\frac{1}{\mu_I} \frac{1}{\phi} \left(\alpha_{t+1} \frac{\delta\rho y}{\phi} - 1 + \frac{1}{\phi} S_I \right)}{[1-\sigma_I\delta R_{t+1}]^2} \\ \frac{\partial H}{\partial S_I} &= \frac{1}{\mu_I} \frac{1}{\phi} (1-\sigma_E) \rho(1-\alpha)y + \sigma_I \frac{1}{\mu_I} \left(\alpha \frac{\delta\rho y}{\phi} - 1 + \frac{1}{\phi} S_I \right) \\ \frac{\partial J}{\partial \alpha} &= -\frac{1}{\mu_E} \delta\rho y \frac{1-\sigma_I}{1-\sigma_E\delta} \frac{R_{t+1}}{1-\sigma_I\delta R_{t+1}} \\ \frac{\partial H}{\partial \alpha} &= \frac{1}{\mu_I} (1-\sigma_E) \rho y \left[\frac{\delta\rho y}{\phi} (1-\alpha) - \left(\alpha \frac{\delta\rho y}{\phi} - 1 + \frac{1}{\phi} S_I \right) \right] + \sigma_I \frac{1}{\mu_I} \left(\alpha \frac{\delta\rho y}{\phi} - 1 + \frac{1}{\phi} S_I \right) \delta\rho y. \end{aligned}$$

Moreover, using the adjusted excess supply function $\Psi(\alpha, S_I, \sigma_E, \sigma_I)$ it is straightforward to show that $\partial H/\partial \alpha > 0$ for the high steady state equilibrium. Thus,

$$\begin{aligned} \frac{dn_E^*(S_I)}{dS_I} &= \frac{1}{\mu_E} \delta\rho(1-\alpha)y \frac{1-\sigma_I}{1-\sigma_E\delta} \frac{\frac{1}{\mu_I} \frac{1}{\phi} Z_{t+1}}{[1-\sigma_I\delta R_{t+1}]^2} \\ &\quad + \frac{\left[\frac{1}{\mu_I} \frac{1}{\phi} (1-\sigma_E) \rho(1-\alpha)y + \sigma_I \frac{1}{\mu_I} Z \right] \frac{1}{\mu_E} \delta\rho y \frac{1-\sigma_I}{1-\sigma_E\delta} \frac{R_{t+1}}{1-\sigma_I\delta R_{t+1}}}{\frac{\partial H}{\partial \alpha}}, \end{aligned}$$

where

$$Z = \alpha \frac{\delta\rho y}{\phi} - 1 + \frac{1}{\phi} S_I \quad Z_{t+1} = \alpha_{t+1} \frac{\delta\rho y}{\phi} - 1 + \frac{1}{\phi} S_I.$$

Note that $n_E^*(S_I = 0) = n_E^*(S_E = 0) = n_E^*$. Thus, we have $n_E^*(S_I) > n_E^*(S_E)$ for all $S_I = S_E > 0$ if $dn_E^*(S_I)/dS_I > dn_E^*(S_E)/dS_E$, which is equivalent to

$$\begin{aligned} & \delta\rho(1-\alpha)y \frac{1-\sigma_I}{1-\sigma_E\delta} \frac{\frac{1}{\mu_I}\frac{1}{\phi}Z_{t+1}}{[1-\sigma_I\delta R_{t+1}]^2} \frac{\partial H}{\partial\alpha} \\ & + \left[\frac{1}{\mu_I}\frac{1}{\phi} (1-\sigma_E)\rho(1-\alpha)y + \sigma_I\frac{1}{\mu_I}Z \right] \delta\rho y \frac{1-\sigma_I}{1-\sigma_E\delta} \frac{R_{t+1}}{1-\sigma_I\delta R_{t+1}} > \frac{\partial H}{\partial\alpha}. \end{aligned}$$

Using $\partial H/\partial\alpha$ and simplifying we can write this condition as

$$\begin{aligned} & \delta\rho(1-\alpha)y \frac{1-\sigma_I}{1-\sigma_E\delta} \frac{\frac{1}{\mu_I}\frac{1}{\phi}Z_{t+1}}{[1-\sigma_I\delta R_{t+1}]^2} \left[(1-\sigma_E) \left[\frac{\delta\rho y}{\phi} (1-\alpha) - Z \right] + \sigma_I Z \delta \right] \\ & + \left[\frac{1}{\phi} (1-\sigma_E)\rho(1-\alpha)y + \sigma_I Z \right] \delta \frac{1-\sigma_I}{1-\sigma_E\delta} \frac{R_{t+1}}{1-\sigma_I\delta R_{t+1}} \\ & > (1-\sigma_E) \left[\frac{\delta\rho y}{\phi} (1-\alpha) - Z \right] + \sigma_I Z \delta. \end{aligned}$$

Rearranging yields

$$\begin{aligned} & \delta\rho(1-\alpha)y \frac{1-\sigma_I}{1-\sigma_E\delta} \frac{\frac{1}{\mu_I}\frac{1}{\phi}Z_{t+1}}{[1-\sigma_I\delta R_{t+1}]^2} \overbrace{\left[(1-\sigma_E) \left[\frac{\delta\rho y}{\phi} (1-\alpha) - Z \right] + \sigma_I Z \delta \right]}{\equiv T_1} \\ & + \delta \left[\frac{1}{\phi} (1-\sigma_E)\rho(1-\alpha)y + \sigma_I Z \right] \underbrace{\left[\frac{1-\sigma_I}{1-\sigma_E\delta} \frac{R_{t+1}}{1-\sigma_I\delta R_{t+1}} - 1 \right]}_{\equiv T_2} > -(1-\sigma_E) \quad (\text{A.16}) \end{aligned}$$

Recall that $\partial H/\partial\alpha > 0$ in the high steady state equilibrium, which implies that $T_1 > 0$. Moreover, we can write T_2 as

$$T_2 = \underbrace{\frac{1-\sigma_I}{1-\sigma_I\delta R_{t+1}}}_{\equiv L_1} \underbrace{\frac{R_{t+1}}{1-\sigma_E\delta}}_{\equiv L_2} - 1.$$

Note that $L_1 > 1$ because $\delta R_{t+1} > 1$. Moreover, we can immediately see that $R_{t+1} > 1$; and because $\sigma_E\delta < 1$ we have $L_2 > 1$. Consequently, $T_2 > 0$, so that condition (A.16) is satisfied. Thus, $dn_E^*(S_I)/dS_I > dn_E^*(S_E)/dS_E$, and therefore $dn_E^*(S_I) > n_E^*(S_E)$ for all $S_I = S_E > 0$.

11 Model with External and Entrepreneurial Angels

With $\tilde{n} > 0$ the high steady state equilibrium is defined by the following entry condition and market clearing condition:

$$n_E = \frac{1}{\mu_E} \delta \rho (1 - \alpha) y \left[1 + \frac{1}{2\mu_I} \left(\alpha_{t+1} \frac{\delta \rho y}{\phi} - 1 \right)^2 \right] \quad (\text{A.17})$$

$$n_E \phi = \frac{1}{\mu_I} \left(\alpha \frac{\delta \rho y}{\phi} - 1 \right) [\rho n_E (1 - \alpha) y + \tilde{n} \tilde{w}]. \quad (\text{A.18})$$

Using the two conditions we define

$$J \equiv \frac{1}{\mu_E} \delta \rho (1 - \alpha) y \left[1 + \frac{1}{2\mu_I} \left(\alpha_{t+1} \frac{\delta \rho y}{\phi} - 1 \right)^2 \right] - n_E = 0$$

$$H \equiv \frac{1}{\mu_I} \left(\alpha \frac{\delta \rho y}{\phi} - 1 \right) \left[\rho (1 - \alpha) y + \frac{1}{n_E} \tilde{n} \tilde{w} \right] - \phi = 0.$$

Applying Cramer's rule we get

$$\frac{d\alpha^*}{d\tilde{n}} = \frac{\begin{vmatrix} -\frac{\partial J}{\partial \tilde{n}} & \frac{\partial J}{\partial n_E} \\ -\frac{\partial H}{\partial \tilde{n}} & \frac{\partial H}{\partial n_E} \end{vmatrix}}{\begin{vmatrix} \frac{\partial J}{\partial \alpha} & \frac{\partial J}{\partial n_E} \\ \frac{\partial H}{\partial \alpha} & \frac{\partial H}{\partial n_E} \end{vmatrix}} = \frac{-\frac{\partial J}{\partial \tilde{n}} \frac{\partial H}{\partial n_E} + \frac{\partial H}{\partial \tilde{n}} \frac{\partial J}{\partial n_E}}{\frac{\partial J}{\partial \alpha} \frac{\partial H}{\partial n_E} - \frac{\partial H}{\partial \alpha} \frac{\partial J}{\partial n_E}},$$

where $\partial J / \partial \tilde{n} = 0$, $\partial J / \partial n_E = -1$, and

$$\frac{\partial H}{\partial \tilde{n}} = \frac{1}{\mu_I} \left(\alpha \frac{\delta \rho y}{\phi} - 1 \right) \frac{1}{n_E} \tilde{w} > 0 \quad \frac{\partial H}{\partial n_E} = -\frac{1}{\mu_I} \left(\alpha \frac{\delta \rho y}{\phi} - 1 \right) \frac{1}{[n_E]^2} \tilde{n} \tilde{w} < 0$$

$$\frac{\partial J}{\partial \alpha} = -\frac{1}{\mu_E} \delta \rho y \left[1 + \frac{1}{2\mu_I} \left(\alpha_{t+1} \frac{\delta \rho y}{\phi} - 1 \right)^2 \right] < 0.$$

Using the adjusted excess supply function $\Psi(\alpha, \tilde{n})$ one can show that $\partial H/\partial\alpha > 0$ for for the high steady state equilibrium. Thus, $d\alpha^*/d\tilde{n} < 0$. Likewise,

$$\frac{dn_E^*}{d\tilde{n}} = \frac{\begin{vmatrix} \frac{\partial J}{\partial\alpha} & -\frac{\partial J}{\partial\tilde{n}} \\ \frac{\partial H}{\partial\alpha} & -\frac{\partial H}{\partial\tilde{n}} \end{vmatrix}}{\begin{vmatrix} \frac{\partial J}{\partial\alpha} & \frac{\partial J}{\partial n_E} \\ \frac{\partial H}{\partial\alpha} & \frac{\partial H}{\partial n_E} \end{vmatrix}} = \frac{-\frac{\partial J}{\partial\alpha} \frac{\partial H}{\partial\tilde{n}} + \frac{\partial H}{\partial\alpha} \frac{\partial J}{\partial\tilde{n}}}{\frac{\partial J}{\partial\alpha} \frac{\partial H}{\partial n_E} - \frac{\partial H}{\partial\alpha} \frac{\partial J}{\partial n_E}} = \frac{\overbrace{-\frac{\partial J}{\partial\alpha}}^{<0} \overbrace{\frac{\partial H}{\partial\tilde{n}}}^{>0}}{\underbrace{\frac{\partial J}{\partial\alpha}}_{<0} \underbrace{\frac{\partial H}{\partial n_E}}_{<0} + \underbrace{\frac{\partial H}{\partial\alpha}}_{>0}} > 0.$$

Next we analyze how the subsidy $S \in \{S_E, S_I\}$ affects the equilibrium equity stake for investors, $\alpha^*(S)$. For this it is convenient to define the total stock of capital as $K = \lambda\rho(1 - \alpha)yn_E + \tilde{n}\tilde{w}$, with $\lambda \in (0, 1)$. We know from Proposition 1 that $d\alpha^*(S_E, \lambda = 0, \tilde{n} > 0)/dS_E > 0$ (only external angels). Moreover, recall from Proposition 3 that $d\alpha^*(S_E, \lambda = 1, \tilde{n} = 0)/dS_E = 0$ (only entrepreneurial angels). This implies that for $\lambda = 1$ and $\tilde{n} > 0$ we have $d\alpha^*(S_E)/dS_E > 0$. Likewise, we know from Proposition 1 that $d\alpha^*(S_I, \lambda = 0, \tilde{n} > 0)/dS_I < 0$, and from Proposition 4 that $d\alpha^*(S_I, \lambda = 1, \tilde{n} = 0)/dS_I < 0$. Consequently, $d\alpha^*(S_I)/dS_I < 0$ for $\lambda = 1$ and $\tilde{n} > 0$.

To compare the effects of the two policies on entry n_E we need to derive expressions for $dn_E^*(S_E)/dS_E$ and $dn_E^*(S_I)/dS_I$. With a founding subsidy S_E the entry condition and the market clearing condition can be written as

$$J \equiv \frac{1}{\mu_E} \left[\delta\rho(1 - \alpha)y \left[1 + \frac{1}{2\mu_I} \left(\alpha_{t+1} \frac{\delta\rho y}{\phi} - 1 \right)^2 \right] + S_E \right] - n_E = 0 \quad (\text{A.19})$$

$$H \equiv \frac{1}{\mu_I} \left(\alpha \frac{\delta\rho y}{\phi} - 1 \right) \left[\rho(1 - \alpha)y + \frac{1}{n_E} \tilde{n}\tilde{w} \right] - \phi = 0. \quad (\text{A.20})$$

Applying Cramer's rule,

$$\frac{dn_E^*(S_E)}{dS_E} = \frac{\begin{vmatrix} \frac{\partial J}{\partial\alpha} & -\frac{\partial J}{\partial S_E} \\ \frac{\partial H}{\partial\alpha} & -\frac{\partial H}{\partial S_E} \end{vmatrix}}{\begin{vmatrix} \frac{\partial J}{\partial\alpha} & \frac{\partial J}{\partial n_E} \\ \frac{\partial H}{\partial\alpha} & \frac{\partial H}{\partial n_E} \end{vmatrix}} = \frac{-\frac{\partial J}{\partial\alpha} \frac{\partial H}{\partial S_E} + \frac{\partial H}{\partial\alpha} \frac{\partial J}{\partial S_E}}{\frac{\partial J}{\partial\alpha} \frac{\partial H}{\partial n_E} - \frac{\partial H}{\partial\alpha} \frac{\partial J}{\partial n_E}},$$

where $\partial J/\partial S_E = 1/\mu_E$, $\partial H/\partial S_E = 0$, $\partial J/\partial n_E = -1$, and

$$\begin{aligned}\frac{\partial H}{\partial n_E} &= -\frac{1}{\mu_I} \left(\alpha \frac{\delta \rho y}{\phi} - 1 \right) \frac{1}{[n_E]^2} \tilde{n} \tilde{w} < 0 \\ \frac{\partial J}{\partial \alpha} &= -\frac{1}{\mu_E} \delta \rho y \left[1 + \frac{1}{2\mu_I} \left(\alpha_{t+1} \frac{\delta \rho y}{\phi} - 1 \right)^2 \right] < 0.\end{aligned}$$

Moreover, using the adjusted excess supply function $\Psi(\alpha, S_E)$ one can again show that $\partial H/\partial \alpha > 0$. Consequently,

$$\frac{dn_E^*(S_E)}{dS_E} = \frac{\overbrace{\frac{\partial H}{\partial \alpha}}^{>0} \frac{1}{\mu_E}}{\frac{1}{\mu_E} \delta \rho y \left[1 + \frac{1}{2\mu_I} Z_{t+1}^2 \right] \frac{1}{\mu_I} Z \frac{1}{[n_E]^2} \tilde{n} \tilde{w} + \underbrace{\frac{\partial H}{\partial \alpha}}_{>0}} > 0,$$

where

$$Z = \alpha \frac{\delta \rho y}{\phi} - 1 > 0 \quad Z_{t+1} = \alpha_{t+1} \frac{\delta \rho y}{\phi} - 1 > 0.$$

Now consider the effect of a funding subsidy S_I . The entry condition and the market clearing condition can be written as

$$J \equiv \frac{1}{\mu_E} \delta \rho (1 - \alpha) y \left[1 + \frac{1}{2\mu_I} \left(\alpha_{t+1} \frac{\delta \rho y}{\phi} - 1 + \frac{1}{\phi} S_I \right)^2 \right] - n_E = 0 \quad (\text{A.21})$$

$$H \equiv \frac{1}{\mu_I} \left(\alpha \frac{\delta \rho y}{\phi} - 1 + \frac{1}{\phi} S_I \right) \left[\rho (1 - \alpha) y + \frac{1}{n_E} \tilde{n} \tilde{w} \right] - \phi = 0. \quad (\text{A.22})$$

Thus,

$$\frac{dn_E^*(S_I)}{dS_I} = \frac{\begin{vmatrix} \frac{\partial J}{\partial \alpha} & -\frac{\partial J}{\partial S_I} \\ \frac{\partial H}{\partial \alpha} & -\frac{\partial H}{\partial S_I} \end{vmatrix}}{\begin{vmatrix} \frac{\partial J}{\partial \alpha} & \frac{\partial J}{\partial n_E} \\ \frac{\partial H}{\partial \alpha} & \frac{\partial H}{\partial n_E} \end{vmatrix}} = \frac{-\frac{\partial J}{\partial \alpha} \frac{\partial H}{\partial S_I} + \frac{\partial H}{\partial \alpha} \frac{\partial J}{\partial S_I}}{\frac{\partial J}{\partial \alpha} \frac{\partial H}{\partial n_E} - \frac{\partial H}{\partial \alpha} \frac{\partial J}{\partial n_E}},$$

where $\partial J/\partial n_E = -1$, and

$$\frac{\partial J}{\partial S_I} = \frac{1}{\mu_E} \delta \rho (1 - \alpha) y \frac{1}{\mu_I} \left(\alpha_{t+1} \frac{\delta \rho y}{\phi} - 1 + \frac{1}{\phi} S_I \right) \frac{1}{\phi} > 0$$

$$\frac{\partial H}{\partial S_I} = \frac{1}{\mu_I} \frac{1}{\phi} \left[\rho (1 - \alpha) y + \frac{1}{n_E} \tilde{n} \tilde{w} \right] > 0$$

$$\frac{\partial H}{\partial n_E} = -\frac{1}{\mu_I} \left(\alpha \frac{\delta \rho y}{\phi} - 1 + \frac{1}{\phi} S_I \right) \frac{1}{[n_E]^2} \tilde{n} \tilde{w} < 0$$

$$\frac{\partial J}{\partial \alpha} = -\frac{1}{\mu_E} \delta \rho y \left[1 + \frac{1}{2\mu_I} \left(\alpha_{t+1} \frac{\delta \rho y}{\phi} - 1 + \frac{1}{\phi} S_I \right)^2 \right] < 0.$$

Moreover, using the adjusted excess supply function $\Psi(\alpha, S_I)$ one can show that $\partial H/\partial \alpha > 0$ at the high steady state equilibrium. Thus,

$$\begin{aligned} \frac{dn_E^*(S_I)}{dS_I} &= \frac{\frac{1}{\mu_E} \delta \rho y \left[1 + \frac{1}{2\mu_I} \left(Z_{t+1} + \frac{1}{\phi} S_I \right)^2 \right] \frac{1}{\mu_I} \frac{1}{\phi} \left[\rho (1 - \alpha) y + \frac{1}{n_E} \tilde{n} \tilde{w} \right]}{\frac{1}{\mu_E} \delta \rho y \left[1 + \frac{1}{2\mu_I} \left(Z_{t+1} + \frac{1}{\phi} S_I \right)^2 \right] \frac{1}{\mu_I} \left(Z + \frac{1}{\phi} S_I \right) \frac{1}{[n_E]^2} \tilde{n} \tilde{w} + \underbrace{\frac{\partial H}{\partial \alpha}}_{>0}} \\ &\quad + \frac{\underbrace{\frac{\partial H}{\partial \alpha}}_{>0} \frac{1}{\mu_E} \delta \rho (1 - \alpha) y \frac{1}{\mu_I} \left(Z_{t+1} + \frac{1}{\phi} S_I \right) \frac{1}{\phi}}{\frac{1}{\mu_E} \delta \rho y \left[1 + \frac{1}{2\mu_I} \left(Z_{t+1} + \frac{1}{\phi} S_I \right)^2 \right] \frac{1}{\mu_I} \left(Z + \frac{1}{\phi} S_I \right) \frac{1}{[n_E]^2} \tilde{n} \tilde{w} + \underbrace{\frac{\partial H}{\partial \alpha}}_{>0}} > 0. \end{aligned}$$

For a comparison we evaluate the two derivatives at $S_E = S_I = 0$. Note that $\alpha^*(S_E) = \alpha^*(S_I)$ for $S_E = S_I \rightarrow 0$, and

$$\left. \frac{\partial H(S_E)}{\partial \alpha} \right|_{S_E=S_I=0} = \left. \frac{\partial H(S_I)}{\partial \alpha} \right|_{S_E=S_I=0} = \frac{1}{\mu_I} \rho y \left[\frac{\delta \rho y}{\phi} (1 - \alpha) + \frac{\delta}{\phi} \frac{1}{n_E} \tilde{n} \tilde{w} - Z \right]. \quad (\text{A.23})$$

Thus, for $S_E = S_I \rightarrow 0$, we can immediately see that $dn_E^*(S_I)/dS_I > dn_E^*(S_E)/dS_E$ if

$$\begin{aligned} & \delta\rho y \left[1 + \frac{1}{2\mu_I} Z_{t+1}^2 \right] \frac{1}{\mu_I} \frac{1}{\phi} \left[\rho(1-\alpha)y + \frac{1}{n_E} \tilde{n}\tilde{w} \right] \\ & + \frac{\partial H(S_E)}{\partial \alpha} \Big|_{S_E=S_I=0} \delta\rho(1-\alpha)y \frac{1}{\mu_I} Z_{t+1} \frac{1}{\phi} > \frac{\partial H(S_E)}{\partial \alpha} \Big|_{S_E=S_I=0}. \end{aligned}$$

Using (A.23) we can see that this condition is always satisfied when

$$\delta\rho y \left[1 + \frac{1}{2\mu_I} Z_{t+1}^2 \right] \frac{1}{\mu_I} \frac{1}{\phi} \left[\rho(1-\alpha)y + \frac{1}{n_E} \tilde{n}\tilde{w} \right] > \frac{1}{\mu_I} \rho y \left[\frac{\delta\rho y}{\phi} (1-\alpha) + \frac{\delta}{\phi} \frac{1}{n_E} \tilde{n}\tilde{w} - Z \right].$$

This condition can be simplified to

$$\frac{1}{2\mu_I} Z_{t+1}^2 \left[\frac{\delta\rho y}{\phi} (1-\alpha) + \frac{\delta}{\phi} \frac{1}{n_E} \tilde{n}\tilde{w} \right] > -Z,$$

which is clearly satisfied. Thus, $dn_E^*(S_I)/dS_I > dn_E^*(S_E)/dS_E$ for $S_E = S_I \rightarrow 0$, so that $n_E^*(S_I) > n_E^*$ for $S_E = S_I \rightarrow 0$.

Finally, for $\tilde{n} > 0$ the low steady state equilibrium $n_{E,t}^L(\alpha^L) = n_{E,t-1}^L(\alpha^L) > 0$ is also defined by the above equilibrium conditions: (A.17) and (A.18) in case of no subsidy, (A.19) and (A.20) in case of a founding subsidy S_E , and (A.21) and (A.22) in case of a funding subsidy S_I . Moreover, the conditions have the same technical properties at the low steady state equilibrium as shown above for the high steady state equilibrium. This includes the corresponding excess supply function $\Psi(\cdot)$, which is also increasing in α at the low steady state equilibrium with $\alpha^* = \alpha^L$. Thus, we get the same comparative statics results as for the high steady state equilibrium as derived above.

12 Catalyst Policies

Suppose the government offers a one-time subsidy $S_{E,t}$ (founding subsidy) or $S_{I,t}$ (funding subsidy) in period t . We now analyze how the one-time subsidies affect the equilibrium outcome in period t and in period $t+1$. For parsimony we analyze the case with $\tilde{n} = 0$; the case with $\tilde{n} > 0$ is equivalent.

We first analyze the effects in period t . With a one-time founding subsidy $S_{E,t}$ the equilibrium is defined by

$$n_{E,t} = \frac{1}{\mu_E} \left[\delta\rho(1-\alpha_t)y \left[1 + \frac{1}{2\mu_I} \left(\alpha_{t+1} \frac{\delta\rho y}{\phi} - 1 \right)^2 \right] + S_{E,t} \right] \quad (\text{A.24})$$

$$n_{E,t}\phi = \frac{1}{\mu_I} \left(\alpha_t \frac{\delta\rho y}{\phi} - 1 \right) \rho n_{E,t-1} (1 - \alpha_{t-1}) y. \quad (\text{A.25})$$

Note that the stock of capital in period t is exogenous. We therefore define $K_t \equiv \rho n_{E,t-1} (1 - \alpha_{t-1}) y$. Combining the two equilibrium conditions we define

$$H \equiv \frac{1}{\mu_E} [\delta\rho(1-\alpha_t)yZ_{t+1} + S_{E,t}] \phi - \frac{1}{\mu_I} \left(\alpha_t \frac{\delta\rho y}{\phi} - 1 \right) K_t = 0,$$

where

$$Z_{t+1} = 1 + \frac{1}{2\mu_I} \left(\alpha_{t+1} \frac{\delta\rho y}{\phi} - 1 \right)^2. \quad (\text{A.26})$$

Using H we get

$$\frac{d\alpha_t^*(S_{E,t})}{dS_{E,t}} = \frac{\frac{1}{\mu_E} \phi}{\frac{1}{\mu_E} \delta\rho y Z_{t+1} \phi + \frac{1}{\mu_I} \frac{\delta\rho y}{\phi} K_t} > 0.$$

Moreover, totally differentiating (A.24) and using the expression for $d\alpha_t^*(S_{E,t})/dS_{E,t}$ we get

$$\frac{dn_{E,t}^*(S_{E,t})}{dS_{E,t}} = \frac{1}{\mu_E} \left[-\delta\rho y \frac{d\alpha_t^*}{dS_{E,t}} Z_{t+1} + 1 \right] = \frac{\frac{1}{\mu_E} \frac{1}{\mu_I} \frac{1}{\phi} K_t}{\frac{1}{\mu_E} Z_{t+1} \phi + \frac{1}{\mu_I} \frac{1}{\phi} K_t} > 0.$$

Now consider the effect of a one-time funding subsidy $S_{I,t}$. The equilibrium is then defined by

$$n_{E,t} = \frac{1}{\mu_E} \delta\rho(1-\alpha_t)y \left[1 + \frac{1}{2\mu_I} \left(\alpha_{t+1} \frac{\delta\rho y}{\phi} - 1 \right)^2 \right] \quad (\text{A.27})$$

$$n_{E,t}\phi = \frac{1}{\mu_I} \left(\alpha_t \frac{\delta\rho y}{\phi} - 1 + S_{I,t} \frac{1}{\phi} \right) K_t. \quad (\text{A.28})$$

Combining the two equilibrium conditions we define

$$J \equiv \frac{1}{\mu_E} \delta\rho(1-\alpha_t)yZ_{t+1}\phi - \frac{1}{\mu_I} \left(\alpha_t \frac{\delta\rho y}{\phi} - 1 + S_{I,t} \frac{1}{\phi} \right) K_t.$$

Using J we get

$$\frac{d\alpha_t^*(S_{I,t})}{dS_{I,t}} = -\frac{\frac{1}{\mu_I} \frac{1}{\phi} K_t}{\frac{1}{\mu_E} \delta \rho y Z_{t+1} \phi + \frac{1}{\mu_I} \frac{\delta \rho y}{\phi} K_t} < 0.$$

Furthermore, using (A.27) and the expression for $d\alpha_t^*(S_{I,t})/dS_{I,t}$,

$$\frac{dn_{E,t}^*(S_{I,t})}{dS_{I,t}} = -\frac{1}{\mu_E} \delta \rho y \frac{d\alpha_t^*(S_{I,t})}{dS_{I,t}} Z_{t+1} = \frac{\frac{1}{\mu_E} \frac{1}{\mu_I} \frac{1}{\phi} K_t Z_{t+1}}{\frac{1}{\mu_E} Z_{t+1} \phi + \frac{1}{\mu_I} \frac{1}{\phi} K_t} > 0.$$

Finally note that $dn_{E,t}^*(S_{I,t})/dS_{I,t} > dn_{E,t}^*(S_{E,t})/dS_{E,t}$ for all $S_{I,t} = S_{E,t}$ if

$$\frac{\frac{1}{\mu_E} \frac{1}{\mu_I} \frac{1}{\phi} K_t Z_{t+1}}{\frac{1}{\mu_E} Z_{t+1} \phi + \frac{1}{\mu_I} \frac{1}{\phi} K_t} > \frac{\frac{1}{\mu_E} \frac{1}{\mu_I} \frac{1}{\phi} K_t}{\frac{1}{\mu_E} Z_{t+1} \phi + \frac{1}{\mu_I} \frac{1}{\phi} K_t},$$

which simplifies to $Z_{t+1} > 1$. We can see from (A.26) that $Z_{t+1} > 1$, so that $dn_{E,t}^*(S_{I,t})/dS_{I,t} > dn_{E,t}^*(S_{E,t})/dS_{E,t}$ for all $S_{I,t} = S_{E,t}$. And because $n_{E,t}^*(S_{I,t} = 0) = n_{E,t}^*(S_{E,t} = 0)$, this implies that $n_{E,t}^*(S_{I,t}) > n_{E,t}^*(S_{E,t})$ for all $S_{I,t} = S_{E,t}$. Moreover, we know that $\alpha_t^*(S_{I,t} = 0) = \alpha_t^*(S_{E,t} = 0)$. And the fact that $d\alpha_t^*(S_{E,t})/dS_{E,t} > 0$ and $d\alpha_t^*(S_{I,t})/dS_{I,t} < 0$ then implies that $\alpha_t^*(S_{I,t}) < \alpha_t^*(S_{E,t})$.

Next we analyze the effects of the catalyst policies in period $t+1$. The equilibrium in period $t+1$ is defined by

$$n_{E,t+1} = \frac{1}{\mu_E} \delta \rho (1 - \alpha_{t+1}) y \left[1 + \frac{1}{2\mu_I} \left(\alpha_{t+2} \frac{\delta \rho y}{\phi} - 1 \right)^2 \right] \quad (\text{A.29})$$

$$n_{E,t+1} \phi = \frac{1}{\mu_I} \left(\alpha_{t+1} \frac{\delta \rho y}{\phi} - 1 \right) \rho n_{E,t}(S_t) (1 - \alpha_t(S_t)) y, \quad (\text{A.30})$$

where $S_t \in \{S_{E,t}, S_{I,t}\}$. For parsimony we define $K_{t+1}(S_t) \equiv \rho n_{E,t}(S_t) (1 - \alpha_t(S_t)) y$, which is the stock of capital in period $t+1$. Recall that $n_{E,t}^*(S_{I,t}) > n_{E,t}^*(S_{E,t})$ and $\alpha_t^*(S_{I,t}) < \alpha_t^*(S_{E,t})$. Thus, $K_{t+1}(S_{I,t}) > K_{t+1}(S_{E,t})$. Combining the two equilibrium conditions we define

$$H \equiv \frac{1}{\mu_E} \delta \rho (1 - \alpha_{t+1}) y Z_{t+1} \phi - \frac{1}{\mu_I} \left(\alpha_{t+1} \frac{\delta \rho y}{\phi} - 1 \right) K_{t+1}(S_t) = 0.$$

Using H we can implicitly differentiate α_{t+1} w.r.t $K_{t+1}(S_t)$:

$$\frac{d\alpha_{t+1}^*(K_{t+1}(S_t))}{dK_{t+1}(S_t)} = -\frac{\frac{1}{\mu_I} \left(\alpha_{t+1} \frac{\delta \rho y}{\phi} - 1 \right)}{\frac{1}{\mu_E} \delta \rho y Z_{t+1} \phi + \frac{1}{\mu_I} \frac{\delta \rho y}{\phi} K_{t+1}(S_t)} < 0.$$

This implies that $\alpha_{t+1}^*(S_{I,t}) < \alpha_{t+1}^*(S_{E,t})$. Moreover, using (A.29),

$$\frac{dn_{E,t+1}^*(K_{t+1}(S_t))}{dK_{t+1}(S_t)} = -\frac{1}{\mu_E} \delta \rho y Z_{t+1} \underbrace{\frac{d\alpha_{t+1}^*(K_{t+1}(S_t))}{dK_{t+1}(S_t)}}_{<0} > 0.$$

Consequently, $n_{E,t+1}^*(S_{I,t}) > n_{E,t+1}^*(S_{E,t})$.

13 Optimal Policy

To consider optimal policies we restrict our attention to non-discriminatory policies, where the government pays the same subsidy to all entrepreneurs or investors. In our discussion of the 'tax equivalence theorem' we have already shown that our funding subsidy can be equivalently modeled as an investment tax credit to investors or to entrepreneurs. The remainder of the proof proceeds in three steps.

Our first step is to show that a funding subsidy S_I generates a higher expected welfare level than a founding subsidy S_E . The second step is to show that in our model there is also an equivalence between investment subsidies and return subsidies. That is, our funding subsidies can be equivalently structured as investment subsidies at the time of investing, or return subsidies in case of success. The final step is to show that, within the confines of our model, there are no other feasible policies that generate higher welfare levels than the funding subsidies. For parsimony we focus on the high steady state equilibrium with $\tilde{n} = 0$; the case with $\tilde{n} > 0$ is equivalent.

13.1 Welfare Comparison: Funding vs. Founding Subsidies

It is convenient to use $S_E = \eta S$ and $S_I = (1 - \eta)S$, with $\eta \in [0, 1]$. Let ψ denote the government's cost of administrating the subsidies; the total cost is then given by $n_E^*(1 + \psi)S$. Moreover, we assume that the government chooses S such that it has a positive effect on the expected welfare.

Using $\hat{\theta} = \alpha \frac{\delta \rho y}{\phi} - 1$ we get the following expression for the total expected utility of all investors, TU_I :

$$\begin{aligned} TU_I &= \rho n_E^* w \left[\int_0^{\hat{\theta} + \frac{1}{\phi}(1-\eta)S} \left(\alpha \frac{\delta \rho y}{\phi} - \theta + \frac{1}{\phi}(1-\eta)S \right) \frac{1}{\mu_I} d\theta + \int_{\hat{\theta} + \frac{1}{\phi}(1-\eta)S}^{\mu_I} \frac{1}{\mu_I} d\theta \right] \\ &= \rho n_E^* w \left[1 + \frac{1}{2\mu_I} \left(\alpha \frac{\delta \rho y}{\phi} - 1 + \frac{1}{\phi}(1-\eta)S \right)^2 \right], \end{aligned}$$

where ρn_E^* is the number of investors in each period, and $w = (1 - \alpha)y$ is the equilibrium wealth of investors in the steady state. Moreover, recall from Proof of Proposition 1 that the total expected utility of all entrepreneurs prior to entry, TU_E , is given by $TU_E = \frac{1}{2}n_E^* \delta \rho (1 - \alpha)y$. We can then write the total steady state welfare function $W = \frac{1}{1-\delta} (TU_E + TU_I - n_E^*(1 + \psi)S)$ as

$$W(\cdot) = \frac{1}{1-\delta} n_E^* \left[\rho(1 - \alpha^*)y \left[\frac{1}{2}\delta + 1 + \frac{1}{2\mu_I} \left(\alpha \frac{\delta \rho y}{\phi} - 1 + \frac{1}{\phi}(1-\eta)S \right)^2 \right] - (1 + \psi)S \right],$$

where $n_E^*(\eta)$ and $\alpha^*(\eta)$ are defined by the following entry and market clearing condition:

$$\begin{aligned} J &\equiv \frac{1}{\mu_E} \left[\delta \rho (1 - \alpha)y \left[1 + \frac{1}{2\mu_I} \left(\alpha_{t+1} \frac{\delta \rho y}{\phi} - 1 + \frac{1}{\phi}(1-\eta)S \right)^2 \right] + \eta S \right] - n_E = 0 \\ H &\equiv \frac{1}{\mu_I} \left(\alpha \frac{\delta \rho y}{\phi} - 1 + \frac{1}{\phi}(1-\eta)S \right) \rho(1 - \alpha)y - \phi = 0. \end{aligned} \tag{A.31}$$

Totally differentiating $W(n_E^*(\eta), \alpha^*(\eta), \eta)$ we get

$$\frac{dW(\cdot)}{d\eta} = \frac{\partial W}{\partial n_E} \frac{dn_E^*(\eta)}{d\eta} + \frac{\partial W}{\partial \alpha} \frac{d\alpha^*(\eta)}{d\eta} + \frac{\partial W}{\partial \eta}.$$

We can immediately see that

$$\begin{aligned} \frac{\partial W}{\partial n_E} &= \frac{1}{1-\delta} \left[\rho(1 - \alpha^*)y \left[\frac{1}{2}\delta + 1 + \frac{1}{2\mu_I} Z^2 \right] - (1 + \psi)S \right] > 0 \\ \frac{\partial W}{\partial \eta} &= -\frac{1}{1-\delta} n_E^* \rho(1 - \alpha^*)y \frac{1}{\mu_I} Z \frac{1}{\phi} S < 0, \end{aligned}$$

where

$$Z = \left(\alpha \frac{\delta \rho y}{\phi} - 1 + \frac{1}{\phi} (1 - \eta) S \right).$$

Moreover,

$$\frac{\partial W}{\partial \alpha} = \frac{1}{1 - \delta} n_E^* \rho y \left[- \left[\frac{1}{2} \delta + 1 + \frac{1}{2 \mu_I} Z^2 \right] + (1 - \alpha^*) \frac{1}{\mu_I} Z \frac{\delta \rho y}{\phi} \right].$$

Note that the market clearing condition (A.31) can be written as $(1 - \alpha) = \mu_I \phi / (\rho y Z)$. Using this expression we get

$$\frac{\partial W}{\partial \alpha} = - \frac{1}{1 - \delta} n_E^* \rho y \left[1 - \frac{1}{2} \delta + \frac{1}{2 \mu_I} Z^2 \right].$$

Because $\delta < 1$, we find that $\partial W / \partial \alpha < 0$.

Next, using Cramer's rule,

$$\frac{dn_E^*(\eta)}{d\eta} = \frac{\begin{vmatrix} -\frac{\partial J}{\partial \eta} & \frac{\partial J}{\partial \alpha} \\ -\frac{\partial H}{\partial \eta} & \frac{\partial H}{\partial \alpha} \end{vmatrix}}{\begin{vmatrix} \frac{\partial J}{\partial n_E} & \frac{\partial J}{\partial \alpha} \\ \frac{\partial H}{\partial n_E} & \frac{\partial H}{\partial \alpha} \end{vmatrix}} = \frac{-\frac{\partial J}{\partial \eta} \frac{\partial H}{\partial \alpha} + \frac{\partial H}{\partial \eta} \frac{\partial J}{\partial \alpha}}{\frac{\partial J}{\partial n_E} \frac{\partial H}{\partial \alpha} - \frac{\partial H}{\partial n_E} \frac{\partial J}{\partial \alpha}},$$

where $\partial J / \partial n_E = -1$, $\partial H / \partial n_E = 0$, and

$$\frac{\partial J}{\partial \eta} = \frac{1}{\mu_E} S \left[1 - \delta \rho (1 - \alpha) y \frac{1}{\mu_I} \left(\alpha_{t+1} \frac{\delta \rho y}{\phi} - 1 + \frac{1}{\phi} (1 - \eta) S \right) \frac{1}{\phi} \right]$$

$$\frac{\partial H}{\partial \eta} = - \frac{1}{\mu_I} \frac{1}{\phi} S \rho (1 - \alpha) y < 0$$

$$\frac{\partial J}{\partial \alpha} = - \frac{1}{\mu_E} \delta \rho y \left[1 + \frac{1}{2 \mu_I} \left(\alpha_{t+1} \frac{\delta \rho y}{\phi} - 1 + \frac{1}{\phi} (1 - \eta) S \right)^2 \right] < 0$$

$$\frac{\partial H}{\partial \alpha} = \frac{1}{\mu_I} \rho y \left[\frac{\delta \rho y}{\phi} (1 - \alpha) - \left(\alpha \frac{\delta \rho y}{\phi} - 1 + \frac{1}{\phi} (1 - \eta) S \right) \right].$$

Furthermore, using the adjusted excess supply function $\Psi(\alpha, \eta)$ one can show that $\partial H/\partial\alpha > 0$. Thus,

$$\frac{dn_E^*(\eta)}{d\eta} = \frac{\frac{\partial J}{\partial\eta} \frac{\partial H}{\partial\alpha} - \frac{\partial H}{\partial\eta} \frac{\partial J}{\partial\alpha}}{\underbrace{\frac{\partial H}{\partial\alpha}}_{>0}}.$$

Clearly, $dn_E^*(\eta)/d\eta < 0$ if the numerator is negative. Note that $\alpha = \alpha_{t+1}$ in the steady state. Thus, using the partial derivatives and simplifying we find that $dn_E^*(\eta)/d\eta < 0$ if

$$\left[1 - \delta\rho(1 - \alpha)y\frac{1}{\mu_I}Z\frac{1}{\phi}\right] \left[\frac{\delta\rho y}{\phi}(1 - \alpha) - Z\right] < \frac{1}{\phi}\rho(1 - \alpha)y\delta \left[1 + \frac{1}{2\mu_I}Z^2\right].$$

Replacing the first $(1 - \alpha)$ in this condition by using the relationship $(1 - \alpha) = \mu_I\phi/(\rho yZ)$, yields

$$(1 - \delta) \left[\frac{\delta\rho y}{\phi}(1 - \alpha) - Z\right] < \frac{1}{\phi}\rho(1 - \alpha)y\delta \left[1 + \frac{1}{2\mu_I}Z^2\right],$$

which can be rearranged to

$$0 < \frac{1}{\phi}\rho(1 - \alpha)y\delta\frac{1}{2\mu_I}Z^2 + \delta\frac{\delta\rho y}{\phi}(1 - \alpha) + (1 - \delta)Z.$$

This condition is clearly satisfied, so that $dn_E^*(\eta)/d\eta < 0$. Likewise,

$$\frac{d\alpha^*(\eta)}{d\eta} = \frac{\begin{vmatrix} \frac{\partial J}{\partial n_E} & -\frac{\partial J}{\partial\eta} \\ \frac{\partial H}{\partial n_E} & -\frac{\partial H}{\partial\eta} \end{vmatrix}}{\begin{vmatrix} \frac{\partial J}{\partial n_E} & \frac{\partial J}{\partial\alpha} \\ \frac{\partial H}{\partial n_E} & \frac{\partial H}{\partial\alpha} \end{vmatrix}} = \frac{-\frac{\partial J}{\partial n_E} \frac{\partial H}{\partial\eta} + \frac{\partial H}{\partial n_E} \frac{\partial J}{\partial\eta}}{\frac{\partial J}{\partial n_E} \frac{\partial H}{\partial\alpha} - \frac{\partial H}{\partial n_E} \frac{\partial J}{\partial\alpha}} = -\frac{\overbrace{\frac{\partial H}{\partial\eta}}^{<0}}{\underbrace{\frac{\partial H}{\partial\alpha}}_{>0}} > 0.$$

To summarize,

$$\frac{dW(\cdot)}{d\eta} = \overbrace{\frac{\partial W}{\partial n_E}}^{>0} \overbrace{\frac{dn_E^*(\eta)}{d\eta}}^{<0} + \overbrace{\frac{\partial W}{\partial\alpha}}^{<0} \overbrace{\frac{d\alpha^*(\eta)}{d\eta}}^{>0} + \overbrace{\frac{\partial W}{\partial\eta}}^{<0}.$$

Thus, $dW(\cdot)/d\eta < 0$, which implies that $\eta^* = 0$. Consequently, the optimal policy is a funding subsidy S_I .

13.2 Equivalence between Investment and Return Subsidies

First consider the benchmark model with only external angels. With a return subsidy each new venture generates the payoff $y + \frac{S_R}{\delta\rho}$. The new market equilibrium is then defined by the following entry and market clearing conditions:

$$n_E = \frac{1}{\mu_E} \delta\rho(1 - \alpha) \left(y + \frac{S_R}{\delta\rho} \right) \quad (\text{A.32})$$

$$n_E \phi = \frac{1}{\mu_I} \left[\alpha \frac{\delta\rho}{\phi} \left(y + \frac{S_R}{\delta\rho} \right) - 1 \right] \tilde{n}\tilde{w}. \quad (\text{A.33})$$

Combining these two conditions we define

$$H \equiv \frac{1}{\mu_E} \delta\rho(1 - \alpha) \left(y + \frac{S_R}{\delta\rho} \right) \phi - \frac{1}{\mu_I} \left[\alpha \frac{\delta\rho}{\phi} \left(y + \frac{S_R}{\delta\rho} \right) - 1 \right] \tilde{n}\tilde{w} = 0,$$

which characterizes the equilibrium equity stake $\alpha^*(S_R)$. Implicitly differentiating $\alpha^*(S_R)$ yields

$$\frac{d\alpha^*(S_R)}{dS_R} = \frac{\frac{1}{\mu_E}(1 - \alpha)\phi - \frac{1}{\mu_I}\alpha\frac{1}{\phi}\tilde{n}\tilde{w}}{\frac{1}{\mu_E}\delta\rho\left(y + \frac{S_R}{\delta\rho}\right)\phi + \frac{1}{\mu_I}\frac{\delta\rho}{\phi}\left(y + \frac{S_R}{\delta\rho}\right)\tilde{n}\tilde{w}}. \quad (\text{A.34})$$

Using H we can derive the following expression for $\frac{1}{\mu_E}(1 - \alpha)\phi$:

$$\frac{1}{\mu_E}(1 - \alpha)\phi = \frac{\frac{1}{\mu_I} \left[\alpha \frac{\delta\rho}{\phi} \left(y + \frac{S_R}{\delta\rho} \right) - 1 \right] \tilde{n}\tilde{w}}{\delta\rho \left(y + \frac{S_R}{\delta\rho} \right)}.$$

Using this expression we get

$$\frac{d\alpha^*(S_R)}{dS_R} = - \frac{\frac{1}{\mu_I}\tilde{n}\tilde{w}}{\delta\rho \left(y + \frac{S_R}{\delta\rho} \right) \left[\frac{1}{\mu_E}\delta\rho \left(y + \frac{S_R}{\delta\rho} \right) \phi + \frac{1}{\mu_I}\frac{\delta\rho}{\phi} \left(y + \frac{S_R}{\delta\rho} \right) \tilde{n}\tilde{w} \right]} < 0.$$

Moreover, using the entry condition (A.32) with (A.34) we get

$$\frac{dn_E^*(S_R)}{dS_R} = \frac{1}{\mu_E} \delta\rho \left[- \frac{d\alpha^*(S_R)}{dS_R} \left(y + \frac{S_R}{\delta\rho} \right) + (1 - \alpha) \frac{1}{\delta\rho} \right] = \frac{\frac{1}{\mu_E} \frac{1}{\mu_I} \frac{1}{\phi} \tilde{n}\tilde{w}}{\frac{1}{\mu_E} \phi + \frac{1}{\mu_I} \frac{1}{\phi} \tilde{n}\tilde{w}} > 0.$$

Using the expression for $dn_E^*(S_I)/dS_I$ as derived in Proof of Proposition 1, we can immediately see that $dn_E^*(S_R)/dS_R = dn_E^*(S_I)/dS_I$. Thus, $n_E^*(S_R) = n_E^*(S_I) = n_E^*(S_E)$ for all $S_R = S_I = S_E$.

We now consider the effect of S_R in the dynamic model, focusing on the high steady state equilibrium. For parsimony we analyze again the case with $\tilde{n} = 0$ (the case with $\tilde{n} > 0$ is equivalent). With a return subsidy $S_R > 0$ the expected utility of an entrepreneur in the high steady state equilibrium is given by

$$U_E(S_R) = \delta\rho(1 - \alpha) \left(y + \frac{S_R}{\delta\rho} \right) \left[\int_0^{\hat{\theta}_{t+1}} \left(\alpha_{t+1} \frac{\delta\rho}{\phi} \left(y + \frac{S_R}{\delta\rho} \right) - \theta \right) \frac{1}{\mu_I} d\theta + \int_{\hat{\theta}_{t+1}}^{\mu_I} \frac{1}{\mu_I} d\theta \right],$$

where $\hat{\theta}_{t+1} = \alpha_{t+1} \frac{\delta\rho}{\phi} \left(y + \frac{S_R}{\delta\rho} \right) - 1$. Thus,

$$U_E(S_R) = \delta\rho(1 - \alpha) \left(y + \frac{S_R}{\delta\rho} \right) \left[1 + \frac{1}{2\mu_I} \left(\alpha_{t+1} \frac{\delta\rho}{\phi} \left(y + \frac{S_R}{\delta\rho} \right) - 1 \right)^2 \right].$$

The high steady state market equilibrium is then defined by the following entry condition and market clearing condition:

$$n_E = \frac{1}{\mu_E} \delta\rho(1 - \alpha) \left(y + \frac{S_R}{\delta\rho} \right) \left[1 + \frac{1}{2\mu_I} \left(\alpha_{t+1} \frac{\delta\rho}{\phi} \left(y + \frac{S_R}{\delta\rho} \right) - 1 \right)^2 \right] \quad (\text{A.35})$$

$$\phi = \frac{1}{\mu_I} \left(\alpha \frac{\delta\rho}{\phi} \left(y + \frac{S_R}{\delta\rho} \right) - 1 \right) \rho(1 - \alpha) \left(y + \frac{S_R}{\delta\rho} \right), \quad (\text{A.36})$$

where (A.36) defines $\alpha^*(S_R)$, and (A.35) then defines $n_E^*(S_R)$. Using (A.36) we define

$$H \equiv \frac{1}{\mu_I} \left(\alpha \frac{\delta\rho}{\phi} \left(y + \frac{S_R}{\delta\rho} \right) - 1 \right) \rho(1 - \alpha) \left(y + \frac{S_R}{\delta\rho} \right) - \phi = 0.$$

Using H we get

$$\frac{d\alpha^*(S_R)}{dS_R} = - \frac{\frac{1}{\mu_I} \rho(1 - \alpha) \left[\alpha \frac{1}{\phi} \left(y + \frac{S_R}{\delta\rho} \right) + \left(\alpha \frac{\delta\rho}{\phi} \left(y + \frac{S_R}{\delta\rho} \right) - 1 \right) \frac{1}{\delta\rho} \right]}{\frac{\partial H}{\partial \alpha}}.$$

Furthermore, using the adjusted excess supply function $\Psi(\alpha, S_R)$ it is straightforward to show that $\partial H/\partial \alpha > 0$ in the high steady state equilibrium. Thus, $d\alpha^*(S_R)/dS_R < 0$. Moreover, totally differentiating (A.35) we find

$$\frac{dn_E^*(S_R)}{dS_R} = - \underbrace{\frac{d\alpha^*(S_R)}{dS_R}}_{<0} \frac{1}{\mu_E} \delta \rho \left(y + \frac{S_R}{\delta \rho} \right) \left[1 + \frac{1}{2\mu_I} \left(\alpha_{t+1} \frac{\delta \rho}{\phi} \left(y + \frac{S_R}{\delta \rho} \right) - 1 \right)^2 \right] + \frac{\partial n_E^*(S_R)}{\partial S_R},$$

where

$$\begin{aligned} \frac{\partial n_E^*(S_R)}{\partial S_R} &= \frac{1}{\mu_E} \delta \rho (1 - \alpha) \frac{1}{\delta \rho} \left[1 + \frac{1}{2\mu_I} \left(\alpha_{t+1} \frac{\delta \rho}{\phi} \left(y + \frac{S_R}{\delta \rho} \right) - 1 \right)^2 \right] \\ &= \frac{1}{\mu_E} \delta \rho (1 - \alpha) \left(y + \frac{S_R}{\delta \rho} \right) \frac{1}{\mu_I} \left(\alpha_{t+1} \frac{\delta \rho}{\phi} \left(y + \frac{S_R}{\delta \rho} \right) - 1 \right) \alpha_{t+1} \frac{1}{\phi} > 0. \end{aligned}$$

Thus, $dn_E^*(S_R)/dS_R > 0$.

Next we show that $n_E^*(S_R) = n_E^*(S_I)$ for all $S_R = S_I$. For this we consider a mix of the two subsidies, $\lambda S_R + (1 - \lambda)S_I$, with $\lambda \in [0, 1]$. The expected utility of an entrepreneur is then given by

$$U_E(S_R, S_I) = \delta \rho (1 - \alpha) \left(y + \lambda \frac{S_R}{\delta \rho} \right) \left[\int_0^{z_{t+1}-1} (z_{t+1} - \theta) \frac{1}{\mu_I} d\theta + \int_{z_{t+1}-1}^{\mu_I} \frac{1}{\mu_I} d\theta \right],$$

where

$$z_{t+1} = \alpha_{t+1} \frac{\delta \rho}{\phi} \left(y + \lambda \frac{S_R}{\delta \rho} \right) + \frac{1}{\phi} (1 - \lambda) S_I.$$

Now define

$$z \equiv \alpha \frac{\delta \rho}{\phi} \left(y + \lambda \frac{S_R}{\delta \rho} \right) + \frac{1}{\phi} (1 - \lambda) S_I,$$

which can be written as

$$\phi z - (1 - \lambda) S_I = \alpha \delta \rho \left(y + \lambda \frac{S_R}{\delta \rho} \right).$$

Using this expression and integrating we get

$$U_E(S_R, S_I) = [\delta \rho y + \lambda S_R + (1 - \lambda) S_I - \phi z] \left[1 + \frac{1}{2\mu_I} (z_{t+1} - 1)^2 \right].$$

The high steady state market equilibrium is then defined by the following entry and market clearing conditions:

$$n_E = \frac{1}{\mu_E} [\delta\rho y + \lambda S_R + (1 - \lambda)S_I - \phi z] \left[1 + \frac{1}{2\mu_I} (z_{t+1} - 1)^2 \right] \quad (\text{A.37})$$

$$\phi = \frac{1}{\mu_I} \left(\alpha \frac{\delta\rho}{\phi} \left(y + \lambda \frac{S_R}{\delta\rho} \right) - 1 + \frac{1}{\phi} (1 - \lambda) S_I \right) \rho (1 - \alpha) \left(y + \lambda \frac{S_R}{\delta\rho} \right). \quad (\text{A.38})$$

Next we show that $dn_E^*(\lambda)/d\lambda = 0$ for $S_R = S_I$. Note that in the steady state equilibrium $\alpha = \alpha_{t+1}$, so that $z = z_{t+1}$. Evaluating the total derivative at $S_R = S_I = S$ we get

$$\left. \frac{dn_E^*(\lambda)}{d\lambda} \right|_{S_R=S_I=S} = \left. \frac{\partial n_E^*(\lambda)}{\partial z} \right|_{S_R=S_I=S} \cdot \left. \frac{dz}{d\lambda} \right|_{S_R=S_I=S} + \left. \frac{\partial n_E^*(\lambda)}{\partial \lambda} \right|_{S_R=S_I=S},$$

where

$$\left. \frac{\partial n_E^*(\lambda)}{\partial \lambda} \right|_{S_R=S_I=S} = \frac{1}{\mu_E} [S - S] \left[1 + \frac{1}{2\mu_I} (z_{t+1} - 1)^2 \right] = 0.$$

Moreover,

$$\left. \frac{dz}{d\lambda} \right|_{S_R=S_I=S} = \frac{1}{\phi} (\delta\rho y + \lambda S) \left. \frac{d\alpha^*(\lambda)}{d\lambda} \right|_{S_R=S_I=S} - \frac{1}{\phi} (1 - \alpha) S.$$

Using (A.38) we get

$$\begin{aligned} \left. \frac{d\alpha^*(\lambda)}{d\lambda} \right|_{S_R=S_I=S} &= - \frac{(1 - \alpha) \left[\left(\alpha \frac{1}{\phi} S - \frac{1}{\phi} S \right) \left(y + \lambda \frac{S}{\delta\rho} \right) + \left(\alpha \frac{\delta\rho}{\phi} \left(y + \lambda \frac{S_R}{\delta\rho} \right) - 1 + \frac{1}{\phi} (1 - \lambda) S \right) \frac{S}{\delta\rho} \right]}{\left(y + \lambda \frac{S}{\delta\rho} \right) \left[\frac{\delta\rho}{\phi} \left(y + \lambda \frac{S}{\delta\rho} \right) (1 - \alpha) - \left(\alpha \frac{\delta\rho}{\phi} \left(y + \lambda \frac{S}{\delta\rho} \right) - 1 + \frac{1}{\phi} (1 - \lambda) S \right) \right]} \\ &= \frac{(1 - \alpha) S \left[\frac{1}{\phi} (1 - \alpha) \left(y + \lambda \frac{S}{\delta\rho} \right) - \left(\alpha \frac{\delta\rho}{\phi} \left(y + \lambda \frac{S_R}{\delta\rho} \right) - 1 + \frac{1}{\phi} (1 - \lambda) S \right) \frac{1}{\delta\rho} \right]}{\delta\rho \left(y + \lambda \frac{S}{\delta\rho} \right) \left[\frac{1}{\phi} \left(y + \lambda \frac{S}{\delta\rho} \right) (1 - \alpha) - \left(\alpha \frac{\delta\rho}{\phi} \left(y + \lambda \frac{S}{\delta\rho} \right) - 1 + \frac{1}{\phi} (1 - \lambda) S \right) \frac{1}{\delta\rho} \right]} \\ &= \frac{(1 - \alpha) S}{\delta\rho y + \lambda S}. \end{aligned}$$

Consequently,

$$\left. \frac{dz}{d\lambda} \right|_{S_R=S_I=S} = \frac{1}{\phi} (\delta\rho y + \lambda S) \frac{(1 - \alpha) S}{\delta\rho y + \lambda S} - \frac{1}{\phi} (1 - \alpha) S = 0,$$

so that $\left. \frac{dn_E^*(\lambda)}{d\lambda} \right|_{S_R=S_I=S} = 0$. Thus, $dn_E^*(S_R)/dS_R = dn_E^*(S_I)/dS_I$ for all $S_R = S_I$, so that $n_E^*(S_R) = n_E^*(S_I)$ for all $S_R = S_I$.

Finally note that an entrepreneur's expected wealth $w^*(\lambda)$ in the high steady state equilibrium is given by

$$w^*(\lambda) = \delta\rho(1 - \alpha) \left(y + \lambda \frac{S_R}{\delta\rho} \right).$$

Evaluating the total derivative of $w^*(\lambda)$ w.r.t. λ at $S_R = S_I = S$ yields

$$\left. \frac{dw^*(\lambda)}{d\lambda} \right|_{S_R=S_I=S} = -(\delta\rho y + \lambda S) \left. \frac{d\alpha^*(\lambda)}{d\lambda} \right|_{S_R=S_I=S} + (1 - \alpha)S = 0.$$

This implies that $w^*(S_R) = w^*(S_I)$ for all $S_R = S_I$.

13.3 Alternative Policies

To identify the set of all feasible non-discriminatory policies, we first consider what states are verifiable for the government to base a subsidy on. The first verifiable action is entry. Our founding subsidies are conditional upon entrepreneurial entry, and by definition apply only to entrepreneurs but not investors. The next verifiable action is investment. Our model already captures investment subsidies to either entrepreneurs or investors. One additional verifiable variable at the investment stage is the investment price α ; we return to this shortly. Finally, the outcome of a venture is verifiable. Our model already captures return subsidies in case of success. It is easy to see that any subsidy to failure would behave equivalently in our model (but would raise concerns about moral hazard in any model extension with private effort).

The only alternative policy to consider here is therefore any subsidy that is contingent on α , as mentioned above. We claim that for any subsidy contingent on α , there exists an equivalent investment subsidy that is not contingent on α (and is therefore already accounted for in our optimality proof). Consider a generic investor subsidy $S_I(\alpha)$, where for simplicity we assume that $S_I(\alpha)$ is weakly monotonous in α (this can be relaxed too). The equilibrium of the model is again characterized by equations (A.21) and (A.22). Consider now the equilibrium level α^* , and the associated subsidy $S_I(\alpha^*)$. Next define a non-contingent subsidy $S_I^* = S_I(\alpha^*)$. It is immediate that S_I^* also satisfies the same equilibrium conditions (A.21) and (A.22) as $S_I(\alpha^*)$. Therefore it achieves the identical market outcome. It follows that any subsidy contingent on α can also be replicated with a subsidy that is not contingent on α . Consequently there cannot be any contingent subsidy that achieves a higher welfare than the non-contingent optimal funding subsidy.