Attainment of Optimal Solution in a Semiobnoxious Location Problem

Emilio Carrizosa
Facultad de Matemáticas, Universidad de Sevilla.
C/ Tarfia, s/n 41012 Sevilla, Spain
ecarriz@cica.es

Dolores Romero-Morales
E.U. Empresariales, Universidad de Cádiz.
C/ Porvera, n^o 54 Jerez (Cádiz), Spain
D.Romero@fac.fbk.eur.nl

Abstract

It is known that if the sum of weights in the Weber problem with attraction and repulsion is positive, then the problem attains an optimal solution.

In this note we extend this result to the nonlinear extension of the abovementioned problem, which has only been addressed in the literature for bounded feasible regions.

Keywords Semiobnoxious facilities, Weber problems, mixed gauges

Studies in Locational Analysis
Issue 12 April 1999 41–48

1 Introduction

The Weber problem [6] has been generalized in several ways to accomodate aspects such as transportation costs not linear in distances or the obnoxious character of the facility, [7, 8]. In this sense, in [3] the so-called Weber problem with attraction and repulsion is addressed,

$$\min_{x \in \mathbb{R}^2} \{ \sum_{a \in A_1} \omega_a \|x - a\| + \sum_{a \in A_2} \omega_a \|x - a\| \} \tag{1}$$

where A_1 and A_2 are disjoint nonempty finite subsets of $\mathbb{R}^2$, the scalars ω_a are negative for those $a \in A_1$ and positive for those $a \in A_2$, and $\| \cdot \|$ is a norm. It is shown there that, if $W = \sum_{a \in A_1 \cup A_2} \omega_a > 0$, then Problem (1) attains an optimal solution, whilst if $W < 0$, then (1) is unbounded.

Problem (1) does not accurately represent real-world situations, since the utility functions involved increase (decrease) linearly with distances.

In [2] one can find a more realistic extension, where the obnoxious character is assumed to have an exponential decay with distances, leading to the problem

$$\min_{x \in S} \{ \sum_{a \in A_1} \omega_a e^{-\|x-a\|} + \sum_{a \in A_2} \omega_a \|x - a\| \} \tag{2}$$

where A_1 and A_2 are disjoint nonempty finite subsets of $\mathbb{R}^2$, $\omega_a > 0$, $a \in A_1 \cup A_2$, S is a bounded polygonal region, and the norm in use is the Euclidean norm.

A further step towards realism appears with [10], where environmental impact and transportation costs are modeled through decreasing (respectively increasing) utility functions with increasing (respectively decreasing) marginal rates. The authors assume that the feasible region is bounded; in practice this always happens, but this excludes particularly simple and important instances such as the classical unconstrained problem, as well as problems in which only constraints defined by forbidden regions are present, namely $S = \{x \in \mathbb{R}^2 \ : \ \|x - a\| \geq R_a, \ a \in A\}$.

In this note we address the problem of attainment of optimal solution in the problem

$$\inf_{x \in S} z(x) \tag{3}$$

where

- $z(x) = \sum_{a \in A} [f_a(\|x - a\|_a^-) + g_a(\|x - a\|_a^+)]$.
- $A \subset \mathbb{R}^n$ is finite and nonempty.

- $\|\cdot\|_a^-$, $\|\cdot\|_a^+$ are gauges in $\mathbb{R}^n$, $a \in A$, [7], possibly different for different points, as proposed, e.g., in [10].
- $f_a : [0, +\infty) \longmapsto \mathbb{R}$ is a nonincreasing lower-semicontinuous convex function.
- $g_a : [0, +\infty) \longmapsto \mathbb{R}$ is a nondecreasing concave (thus lower-semicontinuous, [5]) function.
- S is a nonempty closed, not necessarily bounded, subset of $\mathbb{R}^n$.

Setting $A = A_1 \cup A_2$, $\|\cdot\|_a^- = \|\cdot\|_a^+ = \|\cdot\|$, $f_a(t) = \omega_a\, t$ (respectively $f_a(t) = \omega_a\, e^{-t}$) for all $a \in A_1$ and $f_a(t) = 0$ for all $a \in A_2$, and $g_a(t) = \omega_a\, t$ for all $a \in A_2$ and $g_a(t) = 0$ for all $a \in A_1$, it follows that both Problems (1) and (2) appear as particular instances of Problem (3).

In general, for a point $\hat{a} \in A$ which considers the facility only repulsive (respectively attractive), one should set $g_{\hat{a}} = 0$ (respectively $f_{\hat{a}} = 0$), whislt for points taking into account both effects (i.e., seeing the facility as semiobnoxious), both $f_{\hat{a}}$ and $g_{\hat{a}}$ should be nonzero.

2 Results

We first recall that any pair of gauges $\|\cdot\|_1$ and $\|\cdot\|_2$ in $\mathbb{R}^n$ are equivalent, i.e., one can always find positive constants C_1 and C_2 such that

$$C_1\,\|x\|_1 \leq \|x\|_2 \leq C_2\,\|x\|_1 \quad \forall x \in \mathbb{R}^n. \tag{4}$$

This implies that there exist positive constants C^+, C^- and K_1, K_2 such that, for any $x \in \mathbb{R}^n$ and any $a, b \in A$,

$$\|x\|_a^+ \leq C^+\,\|x\|_b^+ \tag{5}$$
$$\|x\|_a^- \leq C^-\,\|x\|_b^- \tag{6}$$
$$K_1\,\|x\|_a^- \leq \|x\|_a^+ \tag{7}$$
$$K_2\,\|x\|_a^+ \leq \|x\|_a^- \tag{8}$$

We first state two technical lemmata.

Lemma 1 *Let f, $g : [0, +\infty) \longmapsto \mathbb{R}$, with f convex and nonincreasing such that*

$$\lim_{t \uparrow +\infty} [f(t) + g(t)] = +\infty.$$

Then, for any $\Delta \geq 0$ *one has*

$$\lim_{t\uparrow+\infty}[f(t+\Delta)+g(t)]=+\infty.$$

Proof

Since f is convex, the right derivative f'_+ is well-defined and nondecreasing in $(0,+\infty)$. Moreover, f'_+ is bounded from below in $[\hat{t},+\infty)$, for each $\hat{t}>0$.

On the other hand, since f is convex, one has that $f'_+(t)$ is a subgradient of f at t, [9], thus

$$f(t+\Delta)+g(t)\geq f(t)+f'_+(t)\Delta+g(t).$$

Taking limits ($t\uparrow+\infty$) in the right-hand term one gets

$$\lim_{t\uparrow+\infty}[f(t)+f'_+(t)\Delta+g(t)]=+\infty$$

and the result follows. □

As a consequence of Lemma above one has

Lemma 2 *Let* $f,g:[0,+\infty)\longmapsto I\!R$*, with* g *concave and nondecreasing such that*

$$\lim_{t\uparrow+\infty}[f(t)+g(t)]=-\infty.$$

Then, for any $\Delta \geq 0$ *one has*

$$\lim_{t\uparrow+\infty}[f(t)+g(t+\Delta)]=-\infty.$$

Proof

The result follows from Lemma 1 for $\tilde{f}$ and $\tilde{g}$, with $\tilde{f}=-g$, $\tilde{g}=-f$. □

Theorem 1 *Let* C^- *and* K_1 *be positive constants verifying (6) and (7). If*

$$\lim_{t\uparrow+\infty}\sum_{a\in A}[f_a(C^-t)+g_a(K_1t)]=+\infty,$$

the Problem (3) admits an optimal solution, whatever the feasible region S *is.*

Proof

Let $\hat{x} \in S$. Define, for any $b \in A$, the value $\Delta(b) = \max_{a \in A} \|b - a\|_a^-$.

By Lemma 1

$$\lim_{t \uparrow +\infty} \sum_{a \in A} [f_a(C^- t + \Delta(b)) + g_a(K_1 t)] = +\infty.$$

Hence, there exists $t_1(b) > 0$ such that

$$\sum_{a \in A} [f_a(C^- t + \Delta(b)) + g_a(K_1 t)] > z(\hat{x}) \quad \forall t > t_1(b). \tag{9}$$

Define the set $\mathrm{Vor}_-(b)$ as

$$\mathrm{Vor}_-(b) = \{x \in I\!R^n \ : \ \min_{a \in A} \|x - a\|_a^- = \|x - b\|_b^- \}$$

Then for any $x \in \mathrm{Vor}_-(b)$ with $\|x - b\|_b^- > t_1(b)$ one has that

$$\begin{aligned} z(x) &= \sum_{a \in A} [f_a(\|x - a\|_a^-) + g_a(\|x - a\|_a^+)] \\ &\geq \sum_{a \in A} [f_a(\|x - a\|_a^-) + g_a(K_1 \|x - a\|_a^-)] \qquad (10) \\ &\geq \sum_{a \in A} [f_a(\|x - b\|_a^- + \|b - a\|_a^-) + g_a(K_1 \|x - a\|_a^-)] \qquad (11) \\ &\geq \sum_{a \in A} [f_a(C^- \|x - b\|_b^- + \Delta(b)) + g_a(K_1 \|x - b\|_b^-)] \qquad (12) \\ &> z(\hat{x}), \end{aligned}$$

where (10) follows from the nondecreasingness of the g_a's, (11) from the nonincreasingness of the f_a's and finally (12) follows from (9).

Hence, defining B as the closed set

$$B = \bigcup_{b \in A} \{x \in \mathrm{Vor}_-(b) \ : \ \|x - b\|_b^- \leq t_1(b)\},$$

it follows that $\hat{x} \in B$ and any point in $I\!R^n \setminus B$ verifies

$$z(x) > z(\hat{x}). \tag{13}$$

Moreover, since the objective function is assumed to be lower-semicontinuous, there exists $\tilde{x} \in B \cap S$ such that

$$z(x) \geq z(\tilde{x}), \ \forall x \in B \cap S. \tag{14}$$

Hence, by (13) and (14) it follows that for any $x \in S$,

$$z(x) \geq z(\tilde{x}),$$

and the result follows. □

In a similar way, by means of Lemma 2 one can show the following

Theorem 2 *Let C^+ and K_2 be positive constants verifying (5) and (8). If*

$$\lim_{t\uparrow+\infty}\sum_{a\in A}[f_a(K_2t)+g_a(C^+t)]=-\infty,$$

then Problem (3) with unbounded S is unbounded.

Proof

For any $b\in A$, let $\Delta(b)=\max_{a\in A}\|b-a\|_a^-$ and $\mathrm{Vor}_+(b):=\{y\in I\!R^n:\|y-b\|_b^+=\min_{a\in A}\|y-a\|_a^+\}$.

Since S is unbounded there exists a feasible sequence $\{x^k\}\subset S$, such that

$$\|x^k-b\|_b^+\overset{k\mapsto+\infty}{\longmapsto}+\infty\quad\text{for all } b\in A. \tag{15}$$

Since A is a finite set, there exists $\hat{b}\in A$ and a subsequence $\{x^{k_q}\}$ of $\{x^k\}$ such that $x^{k_q}\in\mathrm{Vor}_+(\hat{b})$, for all q.

With a reasoning similar to that used in Theorem 1 one can show that for all $x\in\mathrm{Vor}_+(\hat{b})$,

$$z(x)\le\sum_{a\in A}[f_a(K_2\|x-\hat{b}\|_{\hat{b}}^+)+g_a(C^+\|x-\hat{b}\|_{\hat{b}}^++\Delta(\hat{b}))].$$

By Lemma 2 the second term of the last inequality tends to $+\infty$, when $\|x-b\|_b^+$ tends to $+\infty$. Thus, it follows from (15) that

$$\lim_{q\to+\infty}z(x^{k_q})=-\infty,$$

then Problem (3) has unbounded solution.

□

Since the models addressed so far in the literature, namely Problems (1) and (2), appear as particular instances of Problem (3), Theorems 1 and 2 enable us to rederive the existence theorems for Problems (1) and (2).

Indeed, if $n=2$ and all the gauges are assumed to be the same, taking $C^-=C^+=1$ and $K_1=K_2=1$ one easily deduces the following.

Corollary 1 *[3] If $W:=\sum_{A_1\cup A_2}\omega_a>0$, then Problem (1) admits an optimal solution whatever the feasible region S is. However, if $W<0$, then the problem is unbounded for unbounded S.*

The proof of this result is straightforward if one observes that $(f_a+g_a)(t)=\omega_a t$, $\forall a\in A_1\cup A_2$.

Remark 1 This result can be easily extended to the more realistic case in which, for each $a \in A$, $\|\cdot\|_a^-$ is assumed to be the euclidean distance $\|\cdot\|_2$ (as proposed, e.g., in [1, 4]), but $\|\cdot\|_a^+ = \|\cdot\|_p$, the l_p norm, with $1 \leq p \leq 2$ thus giving a better fit to actual road distances, [6, 7].

It is then straightforward to check that one can take

$$\begin{aligned} C^+ &= C^- = K_1 = 1, \\ K_2 &= \min_{x \neq 0} \frac{\|x\|_2}{\|x\|_p} = 2^{(\frac{1}{2} - \frac{1}{p})}. \end{aligned}$$

It then follows that, if $|\sum_{a \in A_1} \omega_a| < |\sum_{a \in A_2} \omega_a|$ an optimal solution for (1) exists, whilst (1) is unbounded as soon as S is unbounded and

$$|\sum_{a \in A_2} \omega_a| < 2^{(\frac{1}{2} - \frac{1}{p})} \, |\sum_{a \in A_1} \omega_a|.$$

On the other hand, since

$$\lim_{t \uparrow +\infty} \{ \sum_{a \in A_1} \omega_a t + \sum_{a \in A_2} \omega_a e^{-t} \} = +\infty,$$

one obtains

Corollary 2 *Problem (2) has an optimal solution whatever the feasible region S is.*

Acknowledgements. The research of the first author has been partially financed by the Dirección General de Investigación Científica y Técnica, Spain, under grant PB93-0927. This support is gratefully acknowledged.

References

[1] Boffey, T.B., Karkazis, J., (1993), Models and Methods for Location and Routing Decisions Relating to Hazardous Materials, *Studies in Locational Analysis* **5** 149-166.

[2] Chen, P.C., Hansen, P., Jaumard, B., Tuy, H., (1992), Weber's Problem with Attraction and Repulsion, *Journal of Regional Science*, **32**, 467-486.

[3] Drezner, Z., Wesolowsky, G.O., (1991), The Weber Problem on the Plane with some Negative Weights, *INFOR*, **29**, 87-99.

[4] Erkut, E., Neuman, S., (1989), Analytical Models for Locating Undesirable Facilities, *European Journal of Operational Research* **40** 275-291.

[5] Hiriart-Urruty, J.B., Lemaréchal, C., (1993), *Convex Analysis and Minimization Algorithms*, Vol I. Springer-Verlag.

[6] Love, R.F., Morris, J.G., Wesolowsky, G.O., (1988) *Facility Location: Models and methods.* North-Holland.

[7] Plastria, F., (1995), Continuous Location Problems, in Z. Drezner (Ed.), *Facility Location*, Springer-Verlag, 225-262.

[8] Plastria, F., (1995), Continuous Location of Attracting and Undesirable Facilities, *Semi-Plenary Papers Book, EURO XIV Conference*, Jerusalem, 131-148.

[9] Rockafellar, R.T., (1970), *Convex Analysis*, Princeton University Press, Princeton, Nueva Jersey.

[10] Tuy, H., Al-Khayyal, F.A., Zhou, F., (1995), A D.C. Optimization Method for Single Facility Location Problems, *Journal of Global Optimization*, **7**, 209-227.